

# A NONCOMMUTATIVE SIGMA-MODEL

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ABSTRACT. We begin to study a sigma-model in which both the spacetime manifold and the two-dimensional string world-sheet are made noncommutative. We focus on the case where both the spacetime manifold and the two-dimensional string world-sheet are replaced by noncommutative 2-tori. In this situation, we are able to determine when maps between such noncommutative tori exist, to derive the Euler-Lagrange equations, to classify many of the critical points of the Lagrangian, and to study the associated partition function.

## 1. INTRODUCTION

Noncommutative geometry is playing an increasingly important role in physical field theories, especially quantum field theory and string theory. Connes [7] proposed a general formulation of action functionals in noncommutative spacetime, and there is now a large literature on noncommutative field theories (surveyed in part in [11] and [34]). Thus it seems appropriate now to study fully noncommutative sigma-models.

In our previous work [23, 24, 25], we argued that a consistent approach to T-duality for spacetimes  $X$  which are principal torus bundles over another space  $Z$ , with  $X$  possibly equipped with a non-trivial H-flux, forces the consideration of “noncommutative” T-duals in some situations. A special case of this phenomenon was also previously noted by Lowe, Nastase and Ramgoolam [22].

However, this work left open the question of what sort of sigma-model should apply in the situation where the “target space” is no longer a space at all but a noncommutative  $C^*$ -algebra, and in particular (as this is the simplest interesting case), a noncommutative torus.

In classical sigma-models in string theory, the fields are maps  $g: \Sigma \rightarrow X$ , where  $\Sigma$  is closed and 2-dimensional, and the target space  $X$  is 10-dimensional spacetime.

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The leading term in the action is

$$(1) \quad S(g) = \int_{\Sigma} \|\nabla g(x)\|^2 d\sigma(x),$$

where the gradient and norm are computed with respect to suitable Riemannian (or pseudo-Riemannian) metrics on  $\Sigma$  and  $X$ ,  $\sigma$  is volume measure on  $\Sigma$ , and critical points of the action are just harmonic maps  $\Sigma \rightarrow X$ . Usually one adds to (1) a Wess-Zumino term, related to the H-flux, an Einstein term, corresponding to general relativity on  $X$ , and various other terms, but here we will focus on (1) (except in Section 4.2, where the Wess-Zumino term will also come up).

The question we want to treat here is what should replace maps  $g: \Sigma \rightarrow X$  and the action (1) when  $X$  becomes noncommutative. More precisely, we will be interested in the case where we replace  $C_0(X)$ , the algebra of continuous functions on  $X$  vanishing at infinity, by a noncommutative torus. At the end of the paper, we will also comment on what happens in the more complicated case, considered in [23], [24], and [25], where  $A = \Gamma_0(Z, \mathcal{E})$  is the algebra of sections vanishing at infinity of a continuous field  $\mathcal{E}$  of noncommutative 2-tori over a space  $Z$ , which plays the role of reduced or “physically observable” spacetime. (In other words, we think of  $X$  as a bundle over  $Z$  with *noncommutative* 2-torus fibers.)

Naively, since a map  $g: \Sigma \rightarrow X$  is equivalent to a  $C^*$ -algebra morphism  $C_0(X) \rightarrow C(\Sigma)$ , one’s first guess would be to consider  $*$ -homomorphisms  $A \rightarrow C(\Sigma)$ , where  $\Sigma$  is still an ordinary 2-manifold. The problem with this approach when  $A$  is complicated is that often there are no such maps. For example, if  $A = C_0(Z) \otimes A_{\theta}$  with  $\theta$  irrational (this is  $\Gamma_0(Z, \mathcal{E})$  for a trivial field  $\mathcal{E}$  of noncommutative tori over  $Z$ ), then simplicity of  $A_{\theta}$  implies there are *no* non-zero  $*$ -homomorphisms  $A \rightarrow C(\Sigma)$ . Thus the first thing we see is that once spacetime becomes noncommutative, it is necessary to allow the world-sheet  $\Sigma$  to become noncommutative as well.

In most of this paper, we consider a sigma-model based on  $*$ -homomorphisms between noncommutative 2-tori. The first problem is to determine when such maps exist, and this is studied in Section 2. The main result here is Theorem 2.7, which determines necessary and sufficient conditions for existence of a non-zero  $*$ -homomorphism from  $A_{\Theta}$  to  $M_n(A_{\theta})$ , when  $\Theta$  and  $\theta$  are irrational and  $n \geq 1$ . The main section of the paper is Section 3, which studies an energy functional on such  $*$ -homomorphisms. The critical points of the energy are called harmonic maps, and we classify many of them when  $\Theta = \theta$ . We also determine the Euler-Lagrange equations for harmonic maps (Proposition 3.9), which are considerably more complicated than in the commutative case. Subsection 3.3 deals in more detail with the special case of maps from  $C(\mathbb{T}^2)$  to a rational noncommutative torus. Even this case is remarkably complicated, and we discover interesting connections with the field equations studied in [9]. Section 4 deals with various variations on the theory, such as how to incorporate general metrics and the Wess-Zumino term, and what happens when spacetime is a “bundle” of noncommutative tori and not just a single noncommutative torus. Finally, Section 5 discusses what the partition function for our sigma-model may look like.

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## 2. CLASSIFICATION OF MORPHISMS BETWEEN IRRATIONAL ROTATION ALGEBRAS

In principle one should allow replacement of  $\Sigma$  by general noncommutative Riemann surfaces, as defined for example in [28] (in the case of genus 0) and [26] (in the case of genus  $> 1$ ), but since here we take our spacetimes to be noncommutative tori, it is natural to consider the “genus one” case and to replace  $C(\Sigma)$  by  $A_\theta$  for some  $\theta$ . This case was already discussed and studied in [9], but only in the case of exceptionally simple target spaces  $X$ . In fact, in [8] and [9],  $X$  was taken to be  $S^0$ , i.e., the algebra  $A$  was taken to be  $\mathbb{C} \oplus \mathbb{C}$ . (Or alternatively, one could say that they took  $A = \mathbb{C}$ , but allowed non-unital maps.)

We begin by classifying  $*$ -homomorphisms. We begin with the (easy) case of unital maps.

**Theorem 2.1.** *Fix  $\Theta$  and  $\theta$  in  $(0, 1)$ , both irrational. There is a unital  $*$ -homomorphism  $\varphi: A_\Theta \rightarrow A_\theta$  if and only if  $\Theta = c\theta + d$  for some  $c, d \in \mathbb{Z}$ ,  $c \neq 0$ . Such a  $*$ -homomorphism  $\varphi$  can be chosen to be an isomorphism onto its image if and only if  $c = \pm 1$ .*

*Proof.* Remember from [29, 30] that projections in irrational rotation algebras are determined up to unitary equivalence by their traces, that  $K_0(A_\theta)$  is mapped isomorphically to the ordered group  $\mathbb{Z} + \theta\mathbb{Z} \subset \mathbb{R}$  by the unique normalized trace  $\text{Tr}$  on  $A_\theta$ , and that the range of the trace  $\text{Tr}$  on projections from  $A_\theta$  itself is precisely  $(\mathbb{Z} + \theta\mathbb{Z}) \cap [0, 1]$ .

Now a unital  $*$ -homomorphism  $\varphi: A_\Theta \rightarrow A_\theta$  must induce an order-preserving map  $\varphi_*$  of  $K_0$  groups sending the class of the identity to the class of the identity. Since both  $K_0$  groups are identified with dense subgroups of  $\mathbb{R}$ , with the induced order and with the class of the identity represented by the number 1, this map can be identified with the inclusion of a subgroup, with 1 going to 1. So  $\Theta$ , identified with a generator of  $K_0(A_\Theta)$ , must lie in  $\mathbb{Z} + \theta\mathbb{Z}$ , say,  $\Theta = c\theta + d$  for some  $c, d \in \mathbb{Z}$ . That proves necessity of the condition, but sufficiency is easy, since  $A_{c\theta+d} \cong A_{c\theta}$  is the universal  $C^*$ -algebra on two unitaries  $U$  and  $V$  satisfying  $UV = e^{2\pi ic\theta} VU$ , while  $A_\theta$  is the universal  $C^*$ -algebra on two unitaries  $u$  and  $v$  satisfying  $uv = e^{2\pi i\theta} vu$ . So define  $\varphi$  by  $\varphi(U) = u^c$ ,  $\varphi(V) = v$ , and the required condition is satisfied. Note of course that if  $c = \pm 1$ , then the images of  $U$  and  $V$  generate  $A_\theta$  and  $\varphi$  is surjective, whereas if  $|c| \neq 1$ , then  $\varphi_*$  is not surjective (and so  $\varphi$  can't be, either).  $\square$

*Remark 2.2.* With notation as in Theorem 2.1, if  $c = \pm 1$ , it is natural to ask if it follows that *any*  $\varphi$  inducing the isomorphism on  $K_0$  is a  $*$ -isomorphism. The answer is definitely “no.” In fact, by [14, Theorem 7.3], which applies because of [15], for any given possible map  $K_0(A_\Theta) \rightarrow K_0(A_\theta)$ , there is a  $*$ -homomorphism  $A_\Theta \rightarrow A_\theta$  inducing any desired group homomorphism  $\mathbb{Z}^2 \cong K_1(A_\Theta) \rightarrow K_1(A_\theta) \cong \mathbb{Z}^2$ , including the 0-map. In particular,  $A_\theta$  always has proper (i.e., non-invertible) unital

\*-endomorphisms. (To prove this, take  $\Theta = \theta$ , and observe that if the induced map on  $K_1$  is not invertible, then the endomorphism of  $A_\theta$  cannot be invertible.) It is not clear, however, whether or not such endomorphisms constructed using the inductive limit structure of [15] can be chosen to be smooth.

But Kodaka [17, 18] has constructed smooth unital \*-endomorphisms  $\Phi$  of  $A_\theta$ , whose image has nontrivial relative commutant, but only when  $\theta$  is a quadratic irrational of a certain type. For a slight improvement on his result, see Theorem 3.7 below.

Note that the de la Harpe-Skandalis determinant  $\Delta$  [10], with the defining property

$$\Delta(e^y) = \frac{\text{Tr}(y)}{2\pi i} \pmod{\mathbb{Z} + \theta\mathbb{Z}},$$

maps the abelianization of the connected component of the identity in the unitary group of  $A_\theta$  to  $\mathbb{C}^\times/(\mathbb{Z} + \theta\mathbb{Z})$ . Thomsen [35] has proved that everything in the kernel of  $\Delta$  is a *finite product* of commutators. But for the element  $e^{2\pi i\theta} \in \ker \Delta$ , we get a stronger result. Since (by [15, 14])  $A_\theta$  has a proper \*-endomorphism  $\varphi$  inducing the 0-map on  $K_1$ , that means there are two unitaries in  $A_\theta$  (namely,  $\varphi(U)$  and  $\varphi(V)$ ) in the connected component of the identity in the unitary group with commutator  $e^{2\pi i\theta}$ .

As far as \*-automorphisms of  $A_\theta$  are concerned, some structural facts have been obtained by Elliott, Kodaka, and Elliott-Rørørdam [13, 19, 16]. Elliott and Rørørdam [16] showed that  $\overline{\text{Inn}}(A_\theta)$ , the closure of the inner automorphisms, is topologically simple, and that  $\text{Aut}(A_\theta)/\overline{\text{Inn}}(A_\theta) \cong GL(2, \mathbb{Z})$ . However, if one looks instead at *smooth* automorphisms, what one can call *diffeomorphisms*, one sees a different picture. For  $\theta$  satisfying a certain Diophantine condition [13],  $\text{Aut}(A_\theta^\infty)$  is an iterated semidirect product,  $(U(A_\theta^\infty)_0/\mathbb{T}) \rtimes (\mathbb{T}^2 \rtimes SL(2, \mathbb{Z}))$ . This is not true without the Diophantine condition [19], but it may still be that  $\text{Aut}(A_\theta^\infty) = \overline{\text{Inn}}(A_\theta^\infty) \rtimes SL(2, \mathbb{Z})$  for all  $\theta$ . (See Elliott's review of [19] in *MathSciNet*.)

Next we consider \*-homomorphisms that are not necessarily unital. We can attack the problem in two steps. If there is a non-zero \*-homomorphism  $\varphi: A_\Theta \rightarrow M_\ell(A_\theta)$ , not necessarily unital, then  $\varphi(1_{A_\Theta}) = p$  is a self-adjoint projection, and  $\text{im } \varphi \subseteq pM_\ell(A_\theta)p$ , which is an algebra strongly Morita-equivalent to  $A_\theta$ . By [30, Corollary 2.6],  $pM_\ell(A_\theta)p$  must be isomorphic to  $M_n(A_\beta)$  for some  $\beta$  in the orbit of  $\theta$  under the action of  $GL(2, \mathbb{Z})$  on  $\mathbb{R}$  by linear fractional transformations. So we are essentially reduced to the unital case covered in Theorem 2.1, except that we have to allow for the possibility of passage to matrix algebras. (This would be the case even if  $\ell = 1$ , since there is not necessarily any relationship between  $n$  and  $\ell$ .) This modification is covered in the following:

**Theorem 2.3.** *Fix  $\Theta$  and  $\theta$  in  $(0, 1)$ , both irrational, and  $n \in \mathbb{N}$ ,  $n \geq 1$ . There is a unital \*-homomorphism  $\varphi: A_\Theta \rightarrow M_n(A_\theta)$  if and only if  $n\Theta = c\theta + d$  for some  $c, d \in \mathbb{Z}$ ,  $c \neq 0$ . Such a \*-homomorphism  $\varphi$  can be chosen to be an isomorphism onto its image if and only if  $n = 1$  and  $c = \pm 1$ .*

*Proof.* The argument is similar to that for Theorem 2.1, since  $K_0(M_n(A_\theta))$  is again isomorphic (as an ordered group) to  $\mathbb{Z} + \theta\mathbb{Z}$ , but this time the class of the identity

is represented by  $n$ , so that if both  $K_0$  groups are identified with subgroups of  $\mathbb{R}$  in the usual way,  $\varphi_*$  must be multiplication by  $n$ . Hence if  $\varphi$  exists,  $n\Theta \in \mathbb{Z} + \theta\mathbb{Z}$ .

For the other direction, suppose we know that  $n\Theta = c\theta + d$ . We need to construct an embedding of  $A_\Theta$  into a matrix algebra over  $A_\theta$ . By [29, Theorem 4],  $A_\Theta = A_{(c\theta+d)/n}$  is strongly Morita equivalent to  $A_{n/(c\theta+d)}$ , which embeds unitaly into  $A_{1/(c\theta+d)}$  as in the proof of Theorem 2.1, and  $A_{1/(c\theta+d)}$  is Morita equivalent to  $A_{c\theta+d} \cong A_{c\theta}$ , which embeds unitaly into  $A_\theta$ . Stringing things together, we get an embedding of  $A_\Theta$  into a matrix algebra over  $A_\theta$ . (By [29, Proposition 2.1], when two unital  $C^*$ -algebras are Morita equivalent, each one embeds as a corner into a matrix algebra over the other.) So we get a non-zero  $*$ -homomorphism  $A_\Theta \rightarrow M_\ell(A_\theta)$  (not necessarily unital), possibly with  $\ell \neq n$ . The induced map  $\varphi_*$  on  $K_0$  can be identified with an order-preserving homomorphism from  $\mathbb{Z} + \left(\frac{c\theta+d}{n}\right)\mathbb{Z}$  to  $\mathbb{Z} + \theta\mathbb{Z}$ . But in fact we can determine this map precisely, using the fact [29, page 425] that the Morita equivalence from  $A_{(c\theta+d)/n}$  to  $A_{n/(c\theta+d)}$  is associated to multiplication by  $n/(c\theta + d)$ , and the Morita equivalence from  $A_{1/(c\theta+d)}$  to  $A_{c\theta+d}$  is associated to multiplication by  $c\theta + d$ . Thus the composite map  $\varphi_*$  is multiplication by  $n$ , and sends the class of  $1_{A_\Theta}$  to  $n$ , which is the class of  $1_n$  in  $K_0(M_\ell(A_\theta))$ , where necessarily  $\ell \geq n$ . Since (by [30]) projections are determined up to unitary equivalence by their classes in  $K_0$ , we can conjugate by a unitary and arrange for  $\varphi$  to map  $A_\Theta$  unitaly to  $M_n(A_\theta)$ .

For the last statement we use [29, Theorem 3], which says that  $A_\Theta$  can be isomorphic to  $M_n(A_\theta)$  only if  $n = 1$ .  $\square$

We can now reorganize our conclusions in a way that is algebraically more appealing. First, it's helpful in terms of motivation to point out the following purely algebraic lemma, which we suspect is known, though we don't know where to look it up.

**Lemma 2.4.** *Let  $M$  be the submonoid (**not** a subgroup) of  $GL(2, \mathbb{Q})$  consisting of matrices in  $M_2(\mathbb{Z})$  with non-zero determinant, i.e., of integral matrices having inverses that are not necessarily integral. Then  $M$  is generated by  $GL(2, \mathbb{Z})$  and by the matrices of the form  $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$ ,  $r \in \mathbb{Z} \setminus \{0\}$ .*

*Proof.* First we recall that applying an elementary row or column operation to a matrix is the same as pre- or post-multiplying by an elementary matrix of the form  $\begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix}$  or  $\begin{pmatrix} 1 & 0 \\ \star & 1 \end{pmatrix}$ . So it will suffice to show that, given any matrix  $B \in M$ , we can write it as a product of matrices that reduce via elementary row or column operations (over  $\mathbb{Z}$ ) to things of the form  $\begin{pmatrix} \star & 0 \\ 0 & 1 \end{pmatrix}$ . The proof of this is almost the same as for [31, Theorem 2.3.2]. Write  $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ . Since  $B$  is nonsingular,  $b_{11}$  and  $b_{21}$  can't both be 0. Suppose  $b_{j1}$  is the smaller of the two in absolute value (or if the absolute values are the same, choose  $j = 1$ ). Subtracting an integral multiple of the  $j$ -th row from the other row, we can arrange to decrease the minimal absolute value of the elements in the first column. Proceeding this way and using the Euclidean algorithm, we can reduce the first column to either

$\begin{pmatrix} r \\ 0 \end{pmatrix}$  or  $\begin{pmatrix} 0 \\ r \end{pmatrix}$  (with  $r$  the greatest common divisor of the original  $b_{11}$  and  $b_{21}$ ).

Since we can, if necessary, left multiply by the elementary matrix  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , we can assume the first column has been reduced to  $\begin{pmatrix} r \\ 0 \end{pmatrix}$ , and thus that  $B$  has been reduced to the form

$$\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & b_{22} \end{pmatrix} \begin{pmatrix} 1 & b_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & 0 \\ 0 & 1 \end{pmatrix}.$$

And finally,  $\begin{pmatrix} 1 & 0 \\ 0 & b_{22} \end{pmatrix}$  is conjugate to  $\begin{pmatrix} b_{22} & 0 \\ 0 & 1 \end{pmatrix}$  under the elementary matrix  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .  $\square$

*Remark 2.5.* We can relate this back to the proofs of Theorems 2.1 and 2.3. Elements of  $M$  lying in  $GL(2, \mathbb{Z})$  correspond to Morita equivalences of irrational rotation algebras [29, Theorem 4]. Elements of the form  $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$  act on  $\theta$  by multiplication by  $r$ , and correspond to inclusions  $A_{r\theta} \hookrightarrow A_\theta$ . Lemma 2.4 says that general elements of  $M$  are built out of these two cases. This motivates the following Theorem 2.7.

*Remark 2.6.* The appearance of the monoid  $GL(2, \mathbb{Z}) \subset M \subset GL(2, \mathbb{Q})$ , and also the statement of Lemma 2.4, are somewhat reminiscent of the theory of Hecke operators in the theory of modular forms, which also involve the action of the same monoid  $M$  (on  $GL(2, \mathbb{R})/GL(2, \mathbb{Z})$ ).

**Theorem 2.7.** *Fix  $\Theta$  and  $\theta$  in  $(0, 1)$ , both irrational. Then there is a non-zero  $*$ -homomorphism  $\varphi: A_\Theta \rightarrow M_n(A_\theta)$  for some  $n$ , not necessarily unital, if and only if  $\Theta$  lies in the orbit of  $\theta$  under the action of the monoid  $M$  (of Lemma 2.4) on  $\mathbb{R}$  by linear fractional transformations. The possibilities for  $\text{Tr}(\varphi(1_{A_\Theta}))$  are precisely the numbers  $t = c\theta + d > 0$ ,  $c, d \in \mathbb{Z}$  such that  $t\Theta \in \mathbb{Z} + \theta\mathbb{Z}$ . Once  $t$  is chosen,  $n$  can be taken to be any integer  $\geq t$ .*

*Proof.* First suppose  $\varphi$  exists, and let  $p = \varphi(1_{A_\Theta})$ . Then

$$\varphi_*: K_0(A_\Theta) \rightarrow K_0(M_n(A_\theta)) = K_0(A_\theta)$$

must be an injection of ordered groups sending  $1 \in K_0(A_\Theta)$  to  $t = \text{Tr}(p) = c\theta + d \in \mathbb{Z} + \theta\mathbb{Z}$ . Since both groups are dense subgroups of  $\mathbb{R}$ , this map must be multiplication by  $t$  and must send  $\Theta$  to something in  $\mathbb{Z} + \theta\mathbb{Z}$ . So we have  $t\Theta = a\theta + b$  for some  $a, b \in \mathbb{Z}$ , and

$$\Theta = \frac{a\theta + b}{c\theta + d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \theta.$$

The matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  has integer entries, and can't be singular since the numerator and denominator are both non-zero (being  $\text{Tr}(\varphi(q))$  and  $\text{Tr}(\varphi(1))$ , respectively, where  $q$  is a Rieffel projection in  $A_\Theta$  with trace  $\Theta$ ), and  $(a \ b)$  and  $(c \ d)$  can't

be rational multiples of each other (as that would imply  $\Theta$  is rational). Hence  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  lies in  $M$ , and  $t$  is as required. And since  $p \leq 1_n$ ,  $t \leq n$ .

To prove the converse, suppose  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M$  and  $(c\theta + d)\Theta = a\theta + b$ . Let  $t = c\theta + d$  and choose any integer  $n \geq t$ . Since the range of the trace on projections in  $M_n(A_\theta)$  is  $[0, n] \cap (\mathbb{Z} + \theta\mathbb{Z})$ , we can choose a self-adjoint projection  $p \in M_n(A_\theta)$  with  $\text{Tr}(p) = t$ . The subalgebra  $pM_n(A_\theta)p$  of  $M_n(A_\theta)$  is a full corner (since  $A_\theta$  is simple), hence is strongly Morita equivalent to  $A_\theta$ , hence is  $*$ -isomorphic to  $M_k(A_\beta)$  for some  $\beta$  in the orbit of  $GL(2, \mathbb{Z})$  acting on  $\theta$  [30, Corollary 2.6]. In fact, we can compute  $k$  and  $\beta$ ;  $k$  is the (positive) greatest common divisor of  $c$  and  $d$ , and  $\beta$  is obtained by completing the row vector  $(\frac{c}{k} \ \frac{d}{k})$  to a matrix

$$\begin{pmatrix} a' & b' \\ \frac{c}{k} & \frac{d}{k} \end{pmatrix} \in GL(2, \mathbb{Z})$$

and then letting this act on  $\theta$ . By Theorem 2.3, there is a  $*$ -homomorphism  $\varphi: A_\Theta \rightarrow pM_n(A_\theta)p \cong M_k(A_\beta)$  with  $\varphi(1_{A_\Theta}) = p$  if and only if  $k\Theta \in \mathbb{Z} + \beta\mathbb{Z}$ . But, by assumption,

$$k\Theta = k \frac{a\theta + b}{c\theta + d} = \frac{a\theta + b}{\frac{c}{k}\theta + \frac{d}{k}}$$

while

$$\beta = \frac{a'\theta + b'}{\frac{c}{k}\theta + \frac{d}{k}} \quad \text{with} \quad \begin{pmatrix} a' & b' \\ \frac{c}{k} & \frac{d}{k} \end{pmatrix} \in GL(2, \mathbb{Z}).$$

Note that the transpose matrix

$$\begin{pmatrix} a' & \frac{c}{k} \\ b' & \frac{d}{k} \end{pmatrix}$$

also lies in  $GL(2, \mathbb{Z})$ . So we can solve for integers  $r$  and  $s$  such that

$$\begin{pmatrix} a' & \frac{c}{k} \\ b' & \frac{d}{k} \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

That says exactly that

$$\begin{aligned} r\beta + s &= r \cdot \frac{a'\theta + b'}{\frac{c}{k}\theta + \frac{d}{k}} + s \\ &= \frac{r(a'\theta + b') + s(\frac{c}{k}\theta + \frac{d}{k})}{\frac{c}{k}\theta + \frac{d}{k}} \\ &= \frac{(a'r + \frac{c}{k}s)\theta + (b'r + \frac{d}{k}s)}{\frac{c}{k}\theta + \frac{d}{k}} \\ &= \frac{a\theta + b}{\frac{c}{k}\theta + \frac{d}{k}} \\ &= k\Theta, \end{aligned}$$

as required. □

## 3. HARMONIC MAPS BETWEEN NONCOMMUTATIVE TORI

**3.1. The action and some of its minima for maps between noncommutative tori.** In this section we consider the analogue of the action functional (1) in the context of the  $*$ -homomorphisms classified in the last section. For simplicity, consider first of all a unital  $*$ -homomorphism  $\varphi: A_\Theta \rightarrow A_\theta$  as in Theorem 2.1. As before, denote the canonical generators of  $A_\Theta$  and  $A_\theta$  by  $U$  and  $V$ ,  $u$  and  $v$ , respectively. The natural analogue of  $S(g)$  in our situation is

$$(2) \quad \mathcal{L}(\varphi) = \text{Tr} \left( \delta_1(\varphi(U))^* \delta_1(\varphi(U)) + \delta_2(\varphi(U))^* \delta_2(\varphi(U)) \right. \\ \left. + \delta_1(\varphi(V))^* \delta_1(\varphi(V)) + \delta_2(\varphi(V))^* \delta_2(\varphi(V)) \right).$$

(Except for a factor of two, this is the same as the sum of the “energies” of the unitaries  $\varphi(U)$  and  $\varphi(V)$  in  $A_\theta$ , as defined in [32, §5].) Here  $\delta_1$  and  $\delta_2$  are the infinitesimal generators for the “gauge action” of the group  $\mathbb{T}^2$  on  $A_\theta$ . More precisely,  $\delta_1$  and  $\delta_2$  are defined on the smooth subalgebra  $A_\theta^\infty$  by the formulas

$$(3) \quad \delta_1(u) = 2\pi i u, \quad \delta_2(u) = 0, \quad \delta_1(v) = 0, \quad \delta_2(v) = 2\pi i v.$$

The derivations  $\delta_1$  and  $\delta_2$  play the role of measuring partial derivatives in the two coordinate directions in  $A_\theta$  (which, we recall, plays the role of the worldsheet  $\Sigma$ ), the product of an operator with its adjoint has replaced the norm squared, and integration over  $\Sigma$  has been replaced by the trace. Note for example that if  $\Theta = \theta$  and  $\varphi = \text{Id}$ , the identity map, then we obtain

$$\mathcal{L}(\text{Id}) = \text{Tr} \left( \delta_1(u)^* \delta_1(u) + 0 + 0 + \delta_2(v)^* \delta_2(v) \right) = 8\pi^2.$$

More generally, for the  $*$ -automorphism  $\varphi_A: u \mapsto u^p v^q, v \mapsto u^r v^s$ , with  $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbb{Z})$ , we obtain

$$(4) \quad \mathcal{L}(\varphi_A) = \text{Tr} \left( \delta_1(u^p v^q)^* \delta_1(u^p v^q) + \delta_2(u^p v^q)^* \delta_2(u^p v^q) \right. \\ \left. + \delta_1(u^r v^s)^* \delta_1(u^r v^s) + \delta_2(u^r v^s)^* \delta_2(u^r v^s) \right) \\ = 4\pi^2 \left( p^2 + q^2 + r^2 + s^2 \right).$$

**Conjecture 3.1.** *The value (4) of  $\mathcal{L}(\varphi_A)$  is minimal among all  $\mathcal{L}(\varphi)$ ,  $\varphi: A_\theta^\infty \circlearrowleft$  a  $*$ -endomorphism inducing the matrix  $A \in SL(2, \mathbb{Z})$  on  $K_1(A_\theta) \cong \mathbb{Z}^2$ .*

Note that this conjecture is a close relative of [32, Conjecture 5.4], which deals with maps  $C(S^1) \rightarrow A_\theta$  instead of maps  $A_\Theta \rightarrow A_\theta$ . That conjecture said that the multiples of  $u^m v^n$  minimize the energy of the unitaries in their connected components. Since  $\mathcal{L}(\varphi)$  is twice the sum of the energies of  $\varphi(U)$  and  $\varphi(V)$ , [32, Conjecture 5.4] immediately implies the present conjecture. The following results provide support for Conjecture 3.1.

**Theorem 3.2.** *Conjecture 3.1 is true if  $\varphi: A_\theta^\infty \circlearrowleft$  maps  $u$  to a scalar multiple of itself. (In this case,  $p = s = 1$  and  $q = 0$ .) The minimum is achieved precisely when  $\varphi(v) = \lambda u^r v$ ,  $\lambda \in \mathbb{T}$ .*



*Proof.* Let  $\varphi(u) = \mu u$  and  $\varphi(v) = w$ , where  $\mu \in \mathbb{T}$ ,  $w$  is unitary and smooth, and (necessarily)  $uw = e^{2\pi i\theta}wu$ . Since also  $uv = e^{2\pi i\theta}vu$ , it follows that  $wv^*$  is a unitary commuting with  $u$ . Since the  $*$ -subalgebra generated by  $u$  is maximal abelian, that implies that  $w = f(u)v$ , where  $f: \mathbb{T} \rightarrow \mathbb{T}$  is continuous, and the parameter  $r$  is the winding number of  $f$ . Now we compute that  $\delta_1(f(u)v) = 2\pi i f'(u)uv$ ,  $\delta_2(f(u)v) = 2\pi i f(u)v$ , and hence:

$$\begin{aligned}
\mathcal{L}(\varphi) &= \text{Tr}\left(\delta_1(u)^* \delta_1(u) + \delta_2(u)^* \delta_2(u)\right. \\
&\quad \left.+ \delta_1(f(u)v)^* \delta_1(f(u)v) + \delta_2(f(u)v)^* \delta_2(f(u)v)\right) \\
(5) \qquad &= 4\pi^2 \text{Tr}\left(2 + v^*u^* f'(u)^* f'(u)uv\right) \\
&= 4\pi^2 \text{Tr}\left(2 + f'(u)^* f'(u)\right).
\end{aligned}$$

We can pull  $f: \mathbb{T} \rightarrow \mathbb{T}$  back to a function  $[0, 1] \rightarrow \mathbb{R}$  via the covering map  $z = e^{2\pi it}$ , and then the winding number of  $f$  (as a self-map of  $\mathbb{T}$ ) translates into the difference  $f(1) - f(0)$  (for  $f$  defined on  $[0, 1]$ ). The problem of minimizing (5) is thus the same as that of minimizing  $\int_0^1 |f'(t)|^2 dt$  in the class of smooth functions  $f: [0, 1] \rightarrow \mathbb{R}$  with  $f(1) - f(0) = r$ . Since such a function can be written as  $f(t) = f(0) + tr + g(t)$ , with  $g(0) = g(1) = 0$ , and  $f'(t) = r + g'(t)$ , we have

$$\int_0^1 |f'(t)|^2 dt = \int_0^1 (r^2 + 2rg'(t) + g'(t)^2) dt = r^2 + \|g'\|_{L^2}^2 \geq r^2,$$

with equality exactly when  $g' \equiv 0$ , i.e.,  $g$  constant, and thus  $g \equiv 0$  since  $g(0) = 0$ . Thus equality occurs when (going back to the original notation)  $f(u) = \lambda u^r$ , i.e.,  $\varphi(v) = \lambda u^r v$ , for some constant  $\lambda \in \mathbb{T}$ .  $\square$

We now give a complete proof of Conjecture 3.1 for  $*$ -automorphisms, in the case where the Diophantine condition of [13] is satisfied. The same proof works in general modulo a technical point which we will discuss below.

**Theorem 3.3.** *Conjecture 3.1 is true for  $*$ -automorphisms, assuming the Diophantine condition of [13] is satisfied. In other words, if  $\varphi$  is an  $*$ -automorphism of  $A_\theta^\infty$  inducing the map given by  $A \in SL(2, \mathbb{Z})$  on  $K_1(A_\theta)$ , and if  $\theta$  satisfies the Diophantine condition of [13], then*

$$\mathcal{L}(\varphi) \geq \mathcal{L}(\varphi_A),$$

with equality if and only if  $\varphi(u) = \lambda \varphi_A(u)$ ,  $\varphi(v) = \mu \varphi_A(v)$ , for some  $\lambda, \mu \in \mathbb{T}$ .

*Proof.* What we use from [13] is that the hypothesis on  $\theta$  ensures that we can write  $\varphi(u) = \lambda w \varphi_A(u) w^*$ ,  $\varphi(v) = \mu w \varphi_A(v) w^*$ , for some  $\lambda, \mu \in \mathbb{T}$  and for some unitary  $w \in A_\theta^\infty$ . Suppose  $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbb{Z})$ . Since  $\mathcal{L}(\varphi)$  is a sum of four terms, all of which have basically the same form, it will be enough to estimate the first term; the estimate for the other three is precisely analogous. We find that

$$\begin{aligned}
\delta_1(\varphi(u)) &= \delta_1(\lambda w u^p v^q w^*) \\
&= \lambda(\delta_1(w)u^p v^q w^* + w 2\pi i p u^{p-1} v^q w^* + w u^p v^q \delta_1(w)^*),
\end{aligned}$$

so the first term in  $\mathcal{L}(\varphi)$ ,  $\text{Tr}(\delta_1(\varphi(u))^* \delta_1(\varphi(u)))$  is a sum of nine terms, three “principal” terms and six “cross” terms. Note that  $\bar{\lambda}$  in  $\delta_1(\varphi(u))^*$  cancels the  $\lambda$  in  $\delta_1(\varphi(u))$ , so we can ignore the  $\lambda$  altogether. The three principal terms are

$$\begin{aligned}
(6) \quad & \text{Tr} \left( (\delta_1(w) u^p v^q w^*)^* (\delta_1(w) u^p v^q w^*) \right. \\
& \quad + (w 2\pi i p u^p v^q w^*)^* (w 2\pi i p u^p v^q w^*) \\
& \quad \left. + (w u^p v^q \delta_1(w)^*)^* (w u^p v^q \delta_1(w)^*) \right) \\
& = 2 \text{Tr}(\delta_1(w)^* \delta_1(w)) + 4\pi^2 p^2,
\end{aligned}$$

where in the last step we have used (several times) the invariance of the trace under inner automorphisms.

Now consider the six cross-terms. These are

$$\begin{aligned}
(7) \quad & \text{Tr} \left( (\delta_1(w) u^p v^q w^*)^* (w 2\pi i p u^p v^q w^*) \right. \\
& \quad + (\delta_1(w) u^p v^q w^*)^* (w u^p v^q \delta_1(w)^*) \\
& \quad + (w 2\pi i p u^p v^q w^*)^* (\delta_1(w) u^p v^q w^*) \\
& \quad + (w 2\pi i p u^p v^q w^*)^* (w u^p v^q \delta_1(w)^*) \\
& \quad + (w u^p v^q \delta_1(w)^*)^* (\delta_1(w) u^p v^q w^*) \\
& \quad \left. + (w u^p v^q \delta_1(w)^*)^* (w 2\pi i p u^p v^q w^*) \right) \\
& = \text{Tr} \left( 2\pi i p \delta_1(w)^* w + w (u^p v^q)^* \delta_1(w)^* w u^p v^q \delta_1(w)^* \right. \\
& \quad \left. - 2\pi i p w^* \delta_1(w) - 2\pi i p w \delta_1(w)^* \right. \\
& \quad \left. + \delta_1(w) (u^p v^q)^* w^* \delta_1(w) u^p v^q w^* + 2\pi i p \delta_1(w) w^* \right) \\
& = \text{Tr} \left( w (u^p v^q)^* \delta_1(w)^* w u^p v^q \delta_1(w)^* \right. \\
& \quad \left. + \delta_1(w) (u^p v^q)^* w^* \delta_1(w) u^p v^q w^* \right).
\end{aligned}$$

(Note the use of “integration by parts,” [32, Lemma 2.1].) Now we put (6) and (7) together. We obtain

$$\begin{aligned}
\text{Tr}(\delta_1(\varphi(u))^* \delta_1(\varphi(u))) & = 4\pi^2 p^2 + \text{Tr} \left( 2\delta_1(w)^* \delta_1(w) \right. \\
& \quad + w (u^p v^q)^* \delta_1(w)^* w u^p v^q \delta_1(w)^* \\
& \quad \left. + \delta_1(w) (u^p v^q)^* w^* \delta_1(w) u^p v^q w^* \right).
\end{aligned}$$

We make the substitutions  $T = \delta_1(w)^*w$  and  $W = u^pv^q$ . Note that  $W$  is unitary. We obtain

$$\begin{aligned}
\mathrm{Tr}(\delta_1(\varphi(u))^*\delta_1(\varphi(u))) &= 4\pi^2p^2 + \mathrm{Tr}\left(TT^* + T^*T + W^*TWT + T^*W^*T^*W\right) \\
&\quad \text{(using invariance of the trace under cyclic permutations)} \\
&= 4\pi^2p^2 + \mathrm{Tr}\left(T^*WW^*T + T^*WT^*W^* + WTW^*T\right. \\
&\quad \left.+ WTT^*W^*\right) \\
&= 4\pi^2p^2 + \mathrm{Tr}\left((W^*T + T^*W^*)^*(W^*T + T^*W^*)\right) \\
&\geq 4\pi^2p^2.
\end{aligned}$$

Furthermore, equality holds only if  $W^*T + T^*W^* = 0$ , i.e.,  $T = -WT^*W^*$ . Similar estimates with the other three terms in the energy show that  $\mathcal{L}(\varphi) \geq \mathcal{L}(\varphi_A) = 4\pi^2(p^2 + q^2 + r^2 + s^2)$ , with equality only if  $\delta_j(w)^*w = -Ww^*\delta_j(w)W^*$  and  $\delta_j(w)^*w = -W_1w^*\delta_j(w)W_1^*$ , where  $W_1 = u^rv^s$ . (The conditions involving  $W_1$  come from the analysis of the last two terms in  $\mathcal{L}(\varphi)$ , which use the *second* row of the matrix  $A$ .) So if equality holds,  $W$  and  $W_1$  both conjugate  $w^*\delta_j(w)$  to the negative of its adjoint. In particular,  $w^*\delta_j(w)$  commutes with  $W^*W_1$ . But this unitary generates a maximal abelian subalgebra, so  $w^*\delta_j(w)$  is a function  $f$  of  $W^*W_1$ . So  $w^*\delta_j(w) = f(W^*W_1)$  with  $Wf(W^*W_1)W^* = -f(W^*W_1)^*$ . One can check that these equations can be satisfied only if  $f = 0$ . Indeed, we have the commutation relation  $WW_1 = e^{2\pi i\theta}W_1W$ , so

$$W(W^*W_1)^nW^* = (W_1W^*)^n = e^{2\pi in\theta}(W^*W_1)^n.$$

If we expand  $f$  in a Fourier series,  $f(W^*W_1) = \sum_n c_n(W^*W_1)^n$ , then we must have

$$\begin{aligned}
-f(W^*W_1)^* &= -\sum_n \overline{c_n}(W^*W_1)^{-n} = -\sum_n \overline{c_{-n}}(W^*W_1)^n \\
&= \sum_n c_n W(W^*W_1)^nW^* = \sum_n c_n e^{2\pi in\theta}(W^*W_1)^n.
\end{aligned}$$

Equating coefficients gives

$$-\overline{c_{-n}} = c_n e^{2\pi in\theta}, \quad \text{and replacing } n \text{ by } -n, \quad -\overline{c_n} = c_{-n} e^{-2\pi in\theta}.$$

These give

$$-c_{-n} = \overline{c_n} e^{-2\pi in\theta} = -c_{-n} e^{-4\pi in\theta},$$

so all  $c_n$  must vanish for  $n \neq 0$ . Thus  $f$  is a constant equal to its negative, i.e.,  $f = 0$ , so  $\delta_1(w) = 0$  and  $\delta_2(w) = 0$ ,  $w$  is a scalar, and  $\varphi$  differs from  $\varphi_A$  only by a gauge transformation. That completes the proof.  $\square$

*Remark 3.4.* Note that the same proof always shows that  $\mathcal{L}(\varphi_A) \leq \mathcal{L}(\varphi)$  for any  $\varphi$  in the orbit of  $\varphi_A$  under gauge automorphisms and inner automorphisms, and thus, by continuity, under automorphisms in the closure (in the topology of pointwise  $C^\infty$  convergence) of the inner automorphisms. So if the conjecture of Elliott that  $\mathrm{Aut}(A_\theta^\infty) = \overline{\mathrm{Inn}}(A_\theta^\infty) \rtimes SL(2, \mathbb{Z})$  mentioned earlier is true, the Diophantine condition in Theorem 3.3 is unnecessary.

*Remark 3.5.* After the first draft of this paper was written, Hanfeng Li succeeded in proving [32, Conjecture 5.4] and Conjecture 3.1 (in complete generality). His solution is given in the appendix [20].

*Remark 3.6.* Of course, so far we have neglected smooth proper  $*$ -endomorphisms of  $A_\theta$ , which by [17, 18] certainly exist at least for certain quadratic irrational values of  $\theta$ . We do not know if one can construct such endomorphisms to be energy-minimizing. But we can slightly improve the result of [18] as follows.

**Theorem 3.7.** *Suppose  $\theta$  is irrational. Then there is a (necessarily injective) unital  $*$ -endomorphism  $\Phi: A_\theta \rightarrow A_\theta$ , with image  $B \subsetneq A_\theta$  having non-trivial relative commutant and with a conditional expectation of index-finite type from  $A_\theta$  onto  $B$ , if and only if  $\theta$  is a quadratic irrational number. When this is the case,  $\Phi$  can be chosen to be smooth.*

*Proof.* The “only if” direction and the idea behind the “if” direction are both in [18]. We just need to modify his construction as follows. Suppose  $\theta$  is a quadratic irrational. Thus there exist  $a, b, c \in \mathbb{Z}$  with  $a\theta^2 + b\theta + c = 0$ ,  $a \neq 0$ . Choose  $d \in \mathbb{Z}$  with  $0 < a\theta + d < 1$ , and let  $e$  be an orthogonal projection in  $A_\theta$  with trace  $a\theta + d$ . Since

$$(a\theta + d)\theta = a\theta^2 + d\theta = (d - b)\theta - c \in \mathbb{Z} + \theta\mathbb{Z},$$

by Theorem 2.7, there is an injective  $*$ -homomorphism  $\varphi_1: A_\theta \rightarrow A_\theta$  with image  $eA_\theta e$ . Let  $e^\perp = 1 - e$ . Since  $\text{Tr}(1 - e) = -a\theta + 1 - d$  and

$$(-a\theta + 1 - d)\theta = -a\theta^2 + (1 - d)\theta = (1 + b - d)\theta + c \in \mathbb{Z} + \theta\mathbb{Z},$$

there is also an injective  $*$ -homomorphism  $\varphi_2: A_\theta \rightarrow A_\theta$  with image  $e^\perp A_\theta e^\perp$ . Since  $eA_\theta e$  and  $e^\perp A_\theta e^\perp$  are orthogonal,  $\Phi = \varphi_1 + \varphi_2$  is a unital  $*$ -endomorphism of  $A_\theta$  whose image has  $e$  in its relative commutant. It is clear (since  $e$  can be chosen smooth) that  $\Phi$  can be chosen to be smooth. The last part of the argument can be taken more-or-less verbatim from [17]. Let

$$\Psi(x) = \frac{1}{2} \left( exe + e^\perp x e^\perp + \varphi_2(\varphi_1^{-1}(exe)) + \varphi_1(\varphi_2^{-1}(e^\perp x e^\perp)) \right).$$

Then  $\Psi$  is a faithful conditional expectation onto the image of  $\Phi$ , and it has index-finite type as shown in [17, §2].  $\square$

*Remark 3.8.* As pointed out earlier by Kodaka, the endomorphisms constructed in Theorem 3.7 can be constructed to implement a wide variety of maps on  $K_1$ . In fact, one can even choose  $\Phi$  so that  $\Phi_* = 0$  on  $K_1$ , with  $\Phi$  taking both  $u$  and  $v$  to the connected component of the identity in the unitary group. One can see this as follows. The map  $\Phi$  constructed in Theorem 3.7 can be written as  $\iota \circ \Delta$ , where  $\Delta: A_\theta \rightarrow A_\theta \times A_\theta$  is the diagonal map and  $\iota$  is an inclusion of  $A_\theta \times A_\theta$  into  $A_\theta$  (which exists for  $\theta$  a quadratic irrational). Since “block direct sum” agrees with the addition in  $K_1$ , it follows that (in the notation of the proof above)  $\Phi_* = (\varphi_1)_* + (\varphi_2)_*$  on  $K_1$ . One can easily arrange to have  $(\varphi_1)_* = (\varphi_2)_* = \text{Id}$ , which would make  $\Phi_* =$  multiplication by 2. But if  $\varphi_3$  is the automorphism of  $A_\theta$  with  $u \mapsto u^{-1}$ ,  $v \mapsto v^{-1}$  and we replace  $\Phi = \iota \circ \Delta$  by  $\Phi' = \iota \circ (\text{Id} \times \varphi_3) \circ \Delta$ , then since  $(\varphi_3)_* = -1$  on  $K_1$ , we get an endomorphism  $\Phi'$  inducing the 0-map on  $K_1$ .

In fact, one can modify the construction so that  $\Phi_*$  is any desired endomorphism of  $K_1$ . So far we have seen how to get  $\Phi_* = 2$  or  $\Phi_* = 0$ . To get  $\Phi_* = 1$ , use a construction with three blocks. In other words, choose mutually orthogonal projections  $e$  and  $f$  in  $A_\theta$  so that there exist  $*$ -isomorphisms  $\varphi_1$ ,  $\varphi_2$ , and  $\varphi_3$  from  $A_\theta$  onto each of  $eA_\theta e$ ,  $fA_\theta f$ , and  $(1 - e - f)A_\theta(1 - e - f)$ , respectively. (With  $a, b, c, d$  as above, this can be done by choosing  $\text{Tr } e = (a\theta + d)^2$  and  $\text{Tr } f = (a\theta + d)(1 - d - a\theta)$ .) As above, one can arrange to have  $(\varphi_1)_* = (\varphi_2)_* = 1$  on  $K_1$  and  $(\varphi_3)_* = -1$ . So if  $\Phi = \varphi_1 + \varphi_2 + \varphi_3$ ,  $\Phi$  is a unital  $*$ -endomorphism inducing multiplication by  $1 + 1 - 1 = 1$  on  $K_1$ . Other cases can be done similarly.

**3.2. Euler-Lagrange equations.** In Proposition 3.9 below, we determine the Euler-Lagrange equations for the energy functional,  $\mathcal{L}(\varphi)$  in (2). One striking difference with the classical commutative case, is that one cannot get rid of the “integral”  $\text{Tr}$  in the Euler-Lagrange equations whenever  $\theta$  is irrational. In Corollary 3.10, we construct explicit harmonic maps with respect to  $\mathcal{L}$ .

**Proposition 3.9.** *Let  $\mathcal{L}(\varphi)$  denote the energy functional for a unital  $*$ -endomorphism  $\varphi$  of  $A_\theta$ . Then the Euler-Lagrange equations for  $\varphi$  to be a **harmonic map**, that is, a critical point of  $\mathcal{L}$ , are:*

$$0 = \sum_{j=1}^2 \left\{ \text{Tr} (A \delta_j [\varphi(u)^* \delta_j(\varphi(u))]) + \text{Tr} (B \delta_j [\varphi(v)^* \delta_j(\varphi(v))]) \right\}$$

where  $A, B$  are self-adjoint elements in  $A_\theta$ , constrained to satisfy the equation,

$$A - \varphi(v)^* A \varphi(v) = B - \varphi(u)^* B \varphi(u).$$

*Proof.* Consider the 1-parameter family of  $*$ -endomorphisms of  $A_\theta$  defined by

$$\begin{aligned} \varphi_t(u) &= \varphi(u) e^{ih_1(t)} \\ &= \varphi(u) [1 + ith_1'(0) + O(t^2)], \\ \varphi_t(v) &= \varphi(v) e^{ih_2(t)} \\ &= \varphi(v) [1 + ith_2'(0) + O(t^2)], \end{aligned}$$

where  $h_j(t)$ ,  $j = 1, 2$  are 1-parameter families of self-adjoint operators with  $h_1(0) = 0 = h_2(0)$ . Therefore

$$\delta_j(\varphi_t(u)) = \delta_j(\varphi(u)) + it\delta_j(\varphi(u))h_1'(0) + it\varphi(u)\delta_j(h_1'(0)) + O(t^2),$$

and taking adjoints,

$$\delta_j(\varphi_t(u))^* = \delta_j(\varphi(u))^* - ith_1'(0)\delta_j(\varphi(u))^* - it\delta_j(h_1'(0))\varphi(u)^* + O(t^2),$$

and similarly with  $v$  in place of  $u$ ,  $h_2$  in place of  $h_1$ . Using this, the term of order  $t$  in  $\text{Tr}(\delta_j(\varphi_t(u))^* \delta_j(\varphi_t(u)))$  equals

$$\begin{aligned} (8) \quad & i \text{Tr} (\delta_j(h_1'(0)) (\delta_j(\varphi(u))^* \varphi(u) - \varphi(u)^* \delta_j(\varphi(u)))) \\ & = -2i \text{Tr} (\delta_j(h_1'(0)) \varphi(u)^* \delta_j(\varphi(u))). \end{aligned}$$

(Here we used the fact that since  $\varphi(u)$  is unitary,  $\delta_j(\varphi(u))^* \varphi(u) + \varphi(u)^* \delta_j(\varphi(u)) = 0$ .) Because of “integration by parts” [32, Lemma 2.1], equation (8) equals

$$(9) \quad 2i \text{Tr} (h_1'(0) \delta_j [\varphi(u)^* \delta_j(\varphi(u))]).$$

Similarly, we calculate the term of order  $t$  in  $\text{Tr}(\delta_j(\varphi_t(v))^* \delta_j(\varphi_t(v)))$  to be

$$(10) \quad 2i \text{Tr}(h'_2(0) \delta_j[\varphi(v)^* \delta_j(\varphi(v))]).$$

Setting  $A = h'_1(0)$ ,  $B = h'_2(0)$ , we deduce that the Euler-Lagrange equations for  $\mathcal{L}$ , defined by  $0 = \frac{d}{dt} \mathcal{L}(\varphi_t) \Big|_{t=0}$ , are given as in the Proposition.

We next differentiate the constraint equations,

$$\begin{aligned} 0 &= \frac{d}{dt} (\varphi_t(u) \varphi_t(v) - e^{2\pi i \theta} \varphi_t(v) \varphi_t(u)) \Big|_{t=0} \\ &= \varphi(u) h'_1(0) \varphi(v) + \varphi(u) \varphi(v) h'_2(0) - e^{2\pi i \theta} [\varphi(v) h'_2(0) \varphi(u) + \varphi(v) \varphi(u) h'_1(0)]. \end{aligned}$$

Using the fact that  $\varphi$  is a unital  $*$ -endomorphism of  $A_\theta$ , that is,  $\varphi$  satisfies

$$\varphi(u) \varphi(v) = e^{2\pi i \theta} \varphi(v) \varphi(u)$$

we easily see that the constraint equations of the Proposition are also valid.  $\square$

The following is not especially interesting since it is already implied by the stronger result in [20], but it illustrates how one might check this condition in some cases.

**Corollary 3.10.** *If  $\varphi_A$  is the  $*$ -automorphism of  $A_\theta^\infty$  defined by  $\varphi_A(u) = u^p v^q$  and  $\varphi_A(v) = u^r v^s$ , with  $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbb{Z})$ , then  $\varphi_A$  is a critical point of  $\mathcal{L}(\varphi)$ .*

*Proof.* We compute:

$$\begin{aligned} \delta_1(\varphi_A(u)) &= 2\pi i p \varphi_A(u), & \delta_2(\varphi_A(u)) &= 2\pi i q \varphi_A(u), \\ \delta_1(\varphi_A(v)) &= 2\pi i r \varphi_A(v), & \delta_2(\varphi_A(v)) &= 2\pi i s \varphi_A(v). \end{aligned}$$

Therefore

$$\begin{aligned} \varphi_A(u)^* \delta_1(\varphi_A(u)) &= 2\pi i p, \\ \varphi_A(u)^* \delta_2(\varphi_A(u)) &= 2\pi i q, \\ \varphi_A(v)^* \delta_1(\varphi_A(v)) &= 2\pi i r, \\ \varphi_A(v)^* \delta_2(\varphi_A(v)) &= 2\pi i s, \end{aligned}$$

Applying any derivation  $\delta_j$ ,  $j = 1, 2$ , to any of the terms above gives zero, since they are all constants. Therefore  $\varphi_A$  is a critical point of  $\mathcal{L}$ , by the Euler-Lagrange equations in Proposition 3.9.  $\square$

Of course, a major question is to determine how many critical points there are for  $\mathcal{L}$  *aside from* those of the special form  $\varphi_A$ ,  $A \in SL(2, \mathbb{Z})$ .

**3.3. Certain maps between rational noncommutative tori.** In this subsection we investigate certain harmonic maps between *rational* noncommutative tori. This is an exception to our general focus on irrational rotation algebras, but it might shed some light on what seems to be the most difficult case, of (possibly nonunital) maps  $\varphi: A_\Theta \rightarrow M_m(A_\theta)$  implementing a Morita equivalence when

$$\Theta = 1/\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \theta.$$

In effect, we consider this same situation, but in the case where  $\Theta = n > 1$  is a positive integer, so that  $A_\Theta = C(\mathbb{T}^2)$ , the universal  $C^*$ -algebra generated by two commuting unitaries  $U$  and  $V$ . In this case,  $A_\theta = A_{1/n}$  is an algebra of sections of a bundle over  $\mathbb{T}^2$  with fibers  $M_n(\mathbb{C})$ . This bundle is in fact the endomorphism bundle of a complex vector bundle  $V$  over  $\mathbb{T}^2$ , with Chern class  $c_1(V) \equiv 1 \pmod{n}$ . (More generally,  $A_{k/n}$  is the algebra of sections of the endomorphism bundle of a vector bundle of Chern class  $\equiv k \pmod{n}$ ; one can see this, for instance, from the explicit description of the algebra in [3].) If  $u$  and  $v$  are the canonical unitary generators of  $A_{1/n}$ , then  $u^n$  and  $v^n$  are both central, and generate the center of  $A_{1/n}$ , which is isomorphic to  $C(\mathbb{T}^2)$ , the copy of  $\mathbb{T}^2$  here being identified with the spectrum of the algebra  $A_{1/n}$ . Since the normalized trace on  $A_{1/n}$  sends 1 to 1, it takes the value  $\frac{1}{n}$  on rank-one projections  $e$ , which exist in abundance. (The fact that there are lots of global rank-one projections is due to the fact that the Dixmier-Douady invariant of the algebra vanishes.) A choice of  $e$  determines a  $*$ -isomorphism  $\varphi_e$  from  $A_0 = A_n = C(\mathbb{T}^2)$  to  $eA_{1/n}e$ , sending  $U$  to  $eu^n$  and  $V$  to  $ev^n$ . Let us compute the action functional on  $\varphi_e$ .

**Proposition 3.11.** *With notation as above, i.e., with  $e$  a self-adjoint projection in  $A_{1/n}$  and*

$$\varphi_e: C(\mathbb{T}^2) \xrightarrow{\cong} eA_{1/n}e, \quad \varphi_e(U) = eu^n, \quad \varphi_e(V) = ev^n,$$

we have

$$\mathcal{L}(\varphi_e) = 2 \operatorname{Tr} (\delta_1(e)^2 + \delta_2(e)^2 + 4\pi^2 n^2) .$$

Thus, up to a renormalization, this is the same as the action functional on  $e$  as defined in [8, 9]. Thus  $\varphi_e$  is harmonic exactly when  $e$  is harmonic.

*Proof.* We have

$$\delta_1(eu^n) = \delta_1(e)u^n + 2\pi i n e u^n = (\delta_1(e) + 2\pi i n e)u^n \quad \text{and} \quad \delta_2(eu^n) = \delta_2(e)u^n,$$

and similarly for  $ev^n$  (with the roles of  $\delta_1$  and  $\delta_2$  reversed). Since  $u^n$  and  $v^n$  are central, they cancel out when we compute  $(\delta_1(eu^n))^* \delta_1(eu^n)$ , etc., and we obtain

$$\begin{aligned} (\delta_1(eu^n))^* \delta_1(eu^n) &= (\delta_1(e) + 2\pi i n e)^* (\delta_1(e) + 2\pi i n e) \\ &= (\delta_1(e))^2 + 2\pi i n (\delta_1(e)e - e\delta_1(e)) + 4\pi^2 n^2, \\ (\delta_2(eu^n))^* \delta_2(eu^n) &= (\delta_2(e))^2, \\ (\delta_1(ev^n))^* \delta_1(ev^n) &= (\delta_1(e))^2, \\ (\delta_2(ev^n))^* \delta_2(ev^n) &= (\delta_2(e))^2 + 2\pi i n (\delta_2(e)e - e\delta_2(e)) + 4\pi^2 n^2, \end{aligned}$$

and the result follows since the ‘‘cross-terms’’ have vanishing trace.  $\square$

While a complete classification seems difficult, we at least have an existence theorem.

**Theorem 3.12.** *There exist harmonic nonunital  $*$ -isomorphisms  $\varphi_e: C(\mathbb{T}^2) \rightarrow A_{1/n}$ .*

*Proof.* By Proposition 3.11, it suffices to show that  $A_{1/n}$  contains harmonic rank-1 projections. In terms of the realization of  $A_{1/n}$  as  $\Gamma(T^2, \text{End}(V))$ , the sections of the endomorphism bundle of the complex vector bundle  $V$ , this is equivalent to showing that  $\mathbb{P}(V)$ , the  $\mathbb{C}\mathbb{P}^{n-1}$ -bundle over  $\mathbb{T}^2$  whose fiber at a point  $x$  is the projective space of 1-dimensional subspaces of  $V_x$ , has harmonic sections for its natural connection.

One way to prove this is by using holomorphic geometry. Realize  $\mathbb{T}^2$  as an elliptic curve  $E = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$  and  $V$  as a holomorphic bundle. Then a holomorphic section of  $\mathbb{P}(V)$  is certainly harmonic. But a holomorphic section of  $\mathbb{P}(V)$  will exist provided  $V$  has an everywhere non-vanishing holomorphic section  $s$ , since the line through  $s(z)$  is a point of  $\mathbb{P}(V_z)$  varying holomorphically with  $z$ . Since  $n = \text{rank } V > \dim E = 1$ , this is possible by [2, Theorem 2, p. 426], assuming that  $V$  has “sufficient holomorphic sections,” i.e., that there is a holomorphic section through any point in any fiber. The condition of having sufficient sections is weaker than being ample, which we can arrange by changing  $c_1(V)$  to be sufficiently positive (recall that only  $c_1(V) \bmod n$  is fixed, so we have this flexibility).  $\square$

In preparation for Example 3.14 below, it will be useful to give a concrete model for the algebra  $A_{1/n}$ .

**Proposition 3.13.** *Let  $n > 1$ , and let  $\zeta = e^{2\pi i/n}$ . Fix the  $n \times n$  matrices*

$$u_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \ddots \\ 1 & 0 & 0 & \cdots \end{pmatrix}, \quad v_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & \zeta & 0 & \cdots \\ 0 & 0 & \zeta^2 & \cdots \\ 0 & 0 & 0 & \ddots \end{pmatrix},$$

or in other words  $v_0 = \text{diag}(1, \zeta, \zeta^2, \dots, \zeta^{n-1})$ . Then  $A_{1/n}$  can be identified with the algebra of continuous functions  $f: \mathbb{T}^2 \rightarrow M_n(\mathbb{C})$  satisfying the transformation rules

$$\begin{cases} f(\zeta\lambda, \mu) = v_0^{-1}f(\lambda, \mu)v_0, \\ f(\lambda, \zeta\mu) = u_0f(\lambda, \mu)u_0^{-1}. \end{cases}$$

*Proof.* Observe that  $u_0^n = v_0^n = 1$  and that  $u_0v_0 = \zeta v_0u_0$ . It is then easy to see that the most general irreducible representation of  $A_{1/n}$  is equivalent to one of the form  $\pi_{\mu, \lambda}: u \mapsto \mu u_0, v \mapsto \lambda v_0$  for some  $(\mu, \lambda) \in \mathbb{T}^2$ . However, we are “overcounting,” because it is clear that  $v_0^{-1}$  conjugates  $\pi_{\mu, \lambda}$  to  $\pi_{\zeta\mu, \lambda}$ , and  $u_0$  conjugates  $\pi_{\mu, \lambda}$  to  $\pi_{\mu, \zeta\lambda}$ . The spectrum of the algebra  $A_{1/n}$  can thus be identified with the quotient of  $\mathbb{T}^2$  by the action by multiplication by  $n$ -th roots of unity in both coordinates. The result easily follows.  $\square$

*Example 3.14.* We now give a specific example of this situation in which one can write down an explicit harmonic map. We suspect one can do something similar in general, but to make the calculations easier, we restrict to the case  $n = 2$ . Proposition 3.13 describes  $A_{1/2}$  as the algebra of continuous functions



$f: \mathbb{T}^2 \rightarrow M_2(\mathbb{C})$  satisfying

$$(11) \quad \begin{aligned} f(-\lambda, \mu) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} f(\lambda, \mu) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ f(\lambda, -\mu) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} f(\lambda, \mu) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

If we write  $\lambda = e^{i\theta_1}$  and  $\mu = e^{i\theta_2}$ , we can rewrite (11) by thinking of

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

as defined on  $[0, \pi] \times [0, \pi]$ , subject to boundary conditions

$$(12) \quad \begin{aligned} f_{11}(\pi, \theta_2) &= f_{11}(0, \theta_2), & f_{22}(\pi, \theta_2) &= f_{22}(0, \theta_2), \\ f_{12}(\pi, \theta_2) &= -f_{12}(0, \theta_2), & f_{21}(\pi, \theta_2) &= -f_{21}(0, \theta_2), \\ f_{11}(\theta_1, \pi) &= f_{22}(\theta_1, 0), & f_{22}(\theta_1, \pi) &= f_{11}(\theta_1, 0), \\ f_{12}(\theta_1, \pi) &= f_{21}(\theta_1, 0), & f_{21}(\theta_1, \pi) &= f_{12}(\theta_1, 0). \end{aligned}$$

To get a nonunital harmonic map inducing an isomorphism from  $C(\mathbb{T})$  to a nonunital subalgebra of  $A_{1/2}$ , we need by Proposition 3.11 to choose  $f$  satisfying (12) so that for all  $\theta_1$  and  $\theta_2$ ,  $f(\theta_1, \theta_2)$  is self-adjoint with trace 1 and determinant 0, and so that  $f$  is harmonic. The conditions (12) as well as the conditions for  $f$  to be a rank-one projection will be satisfied provided that  $f$  is of the form:

$$(13) \quad f(\theta_1, \theta_2) = \frac{1}{2} \begin{pmatrix} 1 + \cos(g(\theta_1)) \cos \theta_2 & \sin(g(\theta_1)) - i \cos(g(\theta_1)) \sin \theta_2 \\ \sin(g(\theta_1)) + i \cos(g(\theta_1)) \sin \theta_2 & 1 - \cos(g(\theta_1)) \cos \theta_2 \end{pmatrix}$$

with  $g$  real-valued and satisfying the conditions

$$(14) \quad g(0) = -\frac{\pi}{2}, \quad g(\pi) = \frac{\pi}{2}.$$

For  $f$  to be harmonic, we need to make sure it satisfies the Euler-Lagrange equation  $f(\Delta f) = (\Delta f)f$ , which is derived in [9, §4.1]. In the realization of Proposition 3.13, the canonical generators of  $A_{1/2}$  are given by

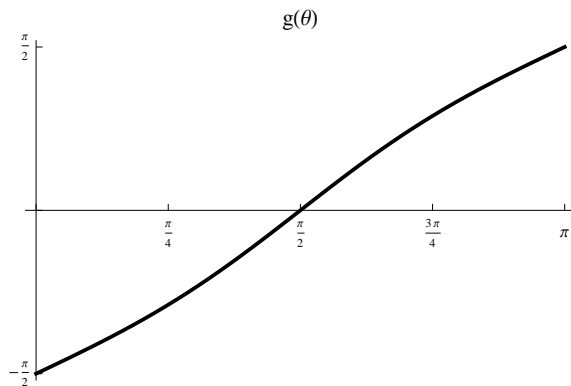
$$u(e^{i\theta_1}, e^{i\theta_2}) = e^{i\theta_1} u_0, \quad v(e^{i\theta_1}, e^{i\theta_2}) = e^{i\theta_2} v_0,$$

so that  $\delta_1$  and  $\delta_2$  act by  $2\pi \frac{\partial}{\partial \theta_1}$  and  $2\pi \frac{\partial}{\partial \theta_2}$ , respectively. Thus up to a factor of  $4\pi^2$ ,  $\Delta$  can be identified with the usual Laplacian in the variables  $\theta_1$  and  $\theta_2$ . A messy calculation, which we performed with *Mathematica*<sup>®</sup>, though one can check it by hand, shows that the commutator of  $f$  and  $\Delta f$  vanishes exactly when the function  $g$  in (13) satisfies the nonlinear (pendulum) differential equation

$$2g''(\theta) + \sin(2g(\theta)) = 0.$$

Subject to the boundary conditions (14), this has a unique solution, which *Mathematica* plots as in Figure 3.14.

Note incidentally that *Mathematica* calculations show that this solution is neither self-dual nor anti-self-dual, in the sense of [9]. In fact, writing out the self-duality and anti-self-duality equations for a projection of the form (13) shows that

FIGURE 1. Plot of  $g(\theta)$  as computed by *Mathematica*

they reduce to  $g'(\theta) = \pm \cos(g(\theta))$ , so the only self-dual or anti-self-dual projections of this form satisfying the initial condition  $g(0) = -\pi/2$  are constant (and thus don't satisfy the other boundary condition in (14)).

It may be of interest to compute the value of  $\mathcal{L}$  for this example. The normalized trace  $\text{Tr}_A$  on  $A_{1/2}$  for matrix-valued functions  $f$  satisfying (12) is

$$\text{Tr}_A f = \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \text{Tr} f(\theta_1, \theta_2) d\theta_1 d\theta_2,$$

so by Proposition 3.11,

$$\begin{aligned} \mathcal{L}(\varphi_f) &= 2 \text{Tr}_A (\delta_1(f)^2 + \delta_2(f)^2 + 4\pi^2 n^2) \text{ with } n = 2 \\ &= 2 \text{Tr}_A \left( 4\pi^2 \left( \frac{\partial f}{\partial \theta_1} \right)^2 + 4\pi^2 \left( \frac{\partial f}{\partial \theta_2} \right)^2 + 4\pi^2 \cdot 4 \right) \\ &= 8\pi^2 \left( 4 + \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \text{Tr} \left( \left( \frac{\partial f}{\partial \theta_1} \right)^2 + \left( \frac{\partial f}{\partial \theta_2} \right)^2 \right) d\theta_1 d\theta_2 \right) \\ &= 8\pi^2 (4 + 0.1116) \approx 32.89\pi^2. \end{aligned}$$

(The integral was computed numerically with *Mathematica*.)

#### 4. VARIATIONS AND REFINEMENTS

One can argue that what we have done up till now was somewhat special, in that we took a very special form for the metric on the “worldsheet,” and ignored the Wess-Zumino term in the action. In this section, we discuss how to generalize the results given earlier in the paper. The modifications to the proofs given in the earlier sections are routine, and most arguments will not be repeated.

**4.1. Spectral triples and sigma-models.** In this subsection, we write a general sigma-model energy functional for spectral triples, that specializes to the cases existing in the literature, including what was discussed earlier in the paper. It is an explicit variant of the discussion in [7, §VI.3] and [9, §2]. Recall that a

spectral triple  $(\mathcal{A}, \mathcal{H}, D)$  is given by an involutive unital algebra  $\mathcal{A}$  represented as bounded operators on a Hilbert space  $\mathcal{H}$  and a self-adjoint operator  $D$  with compact resolvent such that the commutators  $[D, a]$  are bounded for all  $a \in \mathcal{A}$ . A spectral triple  $(\mathcal{A}, \mathcal{H}, D)$  is said to be *even* if the Hilbert space  $\mathcal{H}$  is endowed with a  $\mathbb{Z}_2$ -grading  $\gamma$  which commutes with all  $a \in \mathcal{A}$  and anti-commutes with  $D$ . Suppose in addition that  $(\mathcal{A}, \mathcal{H}, D)$  is  $(2, \infty)$ -summable, which means (assuming for simplicity that  $D$  has no nullspace) that  $\text{Tr}_\omega(a|D|^{-2}) < \infty$ , where  $\text{Tr}_\omega$  denotes the Dixmier trace. We recall from VI.3 in [7] that

$$\psi_2(a_0, a_1, a_2) = \text{Tr}((1 + \gamma)a_0[D, a_1][D, a_2])$$

defines a positive Hochschild 2-cocycle on  $\mathcal{A}$ , where  $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  is the grading operator on  $\mathcal{H}$ , and where  $\text{Tr}$  denotes the Dixmier trace composed with  $D^{-2}$ . In this paper, although we consider the canonical trace  $\text{Tr}$  instead of the above trace, all the properties go through with either choice. Using the Dixmier trace  $\text{Tr}_\omega$  composed with  $D^{-2}$  has the advantage of *scale invariance*, i.e., it is invariant under the replacement of  $D$  by  $\lambda D$  for any nonzero  $\lambda \in \mathbb{C}$ , which becomes relevant when one varies the metric, although for special classes of metrics, the scale invariance can be obtained by other means also. The positivity of  $\psi_2$  means that  $\langle a_0 \otimes a_1, b_0 \otimes b_1 \rangle = \psi_2(b_0^* a_0, a_1, b_1^*)$  defines a positive sesquilinear form on  $\mathcal{A} \otimes \mathcal{A}$ .

We now give a prescription for energy functionals in the sigma-model consisting of homomorphisms  $\varphi: \mathcal{B} \rightarrow \mathcal{A}$ , from a smooth subalgebra of a  $C^*$ -algebra  $\mathcal{B}$  with target the given even  $(2, \infty)$ -summable spectral triple  $(\mathcal{A}, \mathcal{H}, D)$ . Observing that  $\varphi^*(\psi_2)$  is a positive Hochschild 2-cocycle on  $\mathcal{B}$ , we need to choose a formal “metric” on  $\mathcal{B}$ , which is a positive element  $G \in \Omega^2(\mathcal{B})$  in the space of universal 2-forms on  $\mathcal{B}$ . Then evaluation

$$\mathcal{L}_{G,D}(\varphi) = \varphi^*(\psi_2)(G) \geq 0$$

defines a general sigma-model action.

Summarizing, the data for a general sigma-model action consists of

- (1) A  $(2, \infty)$ -summable spectral triple  $(\mathcal{A}, \mathcal{H}, D)$ ;
- (2) A positive element  $G \in \Omega^2(\mathcal{B})$  in the space of universal 2-forms on  $\mathcal{B}$ , known as a metric on  $\mathcal{B}$ .

Consider a unital  $C^*$ -algebra generated by the  $n$  unitaries  $\{U_j : j = 1, \dots, n\}$ , with finitely many relations as in [21], and let  $\mathcal{B}$  be a suitable subalgebra consisting of rapidly vanishing series whose terms are (noncommutative) monomials in the  $U_i$ 's. Then a choice of positive element  $G \in \Omega^2(\mathcal{B})$  (or metric on  $\mathcal{B}$ ) is given by

$$G = \sum_{j,k=1}^n G_{jk} (dU_j)^* dU_k,$$

where the matrix  $(G_{jk})$  is symmetric, real-valued, and positive definite. Then we compute the energy functional in this case,

$$\mathcal{L}_{G,D}(\varphi) = \varphi^*(\psi_2)(G) = \sum_{j,k=1}^n G_{jk} \text{Tr}((1 + \gamma)[D, \varphi(U_j)^*][D, \varphi(U_k)]) \geq 0.$$

The *Euler-Lagrange equations* for  $\varphi$  to be a critical point of  $\mathcal{L}_D$  can be derived as in Proposition 3.9, but since the equations are long, we omit them.

We next give several examples of this sigma-model energy functional. In all of these cases, the target algebra  $\mathcal{A}$  will be  $A_\theta^\infty$ . The first example is the Dąbrowski-Krajewski-Landi model [9], consisting of non-unital  $*$ -homomorphisms  $\varphi: \mathbb{C} \rightarrow A_\theta^\infty$ . Note that  $\varphi(1) = e$  is a projection in the noncommutative torus  $A_\theta$ , and for any  $(2, \infty)$ -summable spectral triple  $(A_\theta^\infty, \mathcal{H}, D)$  on the noncommutative torus, our sigma-model energy functional is

$$\mathcal{L}_D(\varphi) = \text{Tr} [(1 + \gamma)[D, e][D, e]].$$

Choosing the even spectral triple given by  $\mathcal{H} = L^2(A_\theta) \otimes \mathbb{C}^2$  consisting of the Hilbert space closure of  $A_\theta$  in the canonical scalar product coming from the trace, tensored with the 2-dimensional representation space of spinors, and  $D = \gamma_1 \delta_1 + \gamma_2 \delta_2$ , where

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

are the Pauli matrices, we calculate that

$$\mathcal{L}_D(\varphi) = \sum_{j=1}^2 \text{Tr} [(\delta_j e)^2],$$

recovering the action in [9] and the Euler-Lagrange equation  $(\Delta e)e = e(\Delta e)$  there.

Next, we consider the model in Rosenberg [32, §5], consisting of unital  $*$ -homomorphisms  $\varphi: C(S^1) \rightarrow A_\theta^\infty$ . Let  $U$  be the unitary given by multiplication by the coordinate function  $z$  on  $S^1$  (considered as the unit circle  $\mathbb{T}$  in  $\mathbb{C}$ ). Then  $\varphi(U)$  is a unitary in the noncommutative torus  $A_\theta$ , and for any  $(2, \infty)$ -summable spectral triple  $(A_\theta^\infty, \mathcal{H}, D)$  on the noncommutative torus, our sigma-model energy functional is

$$\mathcal{L}_D(\varphi) = \text{Tr} [(1 + \gamma)[D, \varphi(U)^*][D, \varphi(U)]].$$

Choosing the particular spectral triple on the noncommutative torus as above, we calculate that

$$\mathcal{L}_D(\varphi) = \sum_{j=1}^2 \text{Tr} [(\delta_j(\varphi(U)))^* \delta_j(\varphi(U))],$$

recovering the action in [32] and the Euler-Lagrange equation

$$\varphi(U)^* \Delta(\varphi(U)) + (\delta_1(\varphi(U)))^* \delta_1(\varphi(U)) + (\delta_2(\varphi(U)))^* \delta_2(\varphi(U)) = 0$$

there.

The final example is the one treated in this paper. For any (smooth) homomorphism  $\varphi: A_\Theta \rightarrow A_\theta$  and any  $(2, \infty)$ -summable spectral triple  $(A_\theta^\infty, \mathcal{H}, D)$ , and any positive element  $G \in \Omega^2(\mathcal{A}_\Theta)$  (or metric on  $\mathcal{A}_\Theta$ ) given by

$$G = \sum_{j,k=1}^2 G_{ij} (dU_j)^* dU_k,$$

the energy of  $\varphi$  is

$$\mathcal{L}_{G,D}(\varphi) = \varphi^*(\psi_2)(G) = \sum_{j,k=1}^2 G_{jk} \operatorname{Tr}((1 + \gamma)[D, \varphi(U_j)^*][D, \varphi(U_k)]) \geq 0.$$

where  $U, V$  are the canonical generators of  $A_\Theta$ .

Choosing the particular spectral triple on the noncommutative torus as above, we obtain the action and Euler-Lagrange equation considered in §3.

One can consider other choices of spectral triples on  $A_\theta$  defined as follows. For instance, let  $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in M_2(\mathbb{R})$  be a symmetric real-valued positive definite matrix. Then one can consider the 2-dimensional complexified Clifford algebra, with self-adjoint generators  $\gamma_\mu \in M_2(\mathbb{C})$  and relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = g^{\mu\nu}, \quad \mu, \nu = 1, 2,$$

where  $(g^{\mu\nu})$  denotes the matrix  $g^{-1}$ . Then with  $\mathcal{H}$  as before, define  $D = \sum_{\mu=1}^2 \gamma_\mu \delta_\mu$ .

The energy in this more general case is

$$(15) \quad \mathcal{L}_{G,D}(\varphi) = \varphi^*(\psi_2)(G) = \sum_{j,k=1}^2 \sum_{\mu,\nu=1}^2 G_{jk} g^{\mu\nu} \operatorname{Tr}(\delta_\mu(\varphi(U_j))^* \delta_\nu(\varphi(U_k))) \geq 0.$$

In this case, the trace  $\operatorname{Tr}$  is either the Dixmier trace composed with  $D^{-2}$ , or the canonical trace on  $A_\theta$  multiplied by the factor  $\sqrt{\det(g)}$ , to make the energy scale invariant. The Euler-Lagrange equations in this case are an easy modification of those in Proposition 3.9.

**4.2. The Wess-Zumino term.** There is a rather large literature on “noncommutative Wess-Zumino theory” or “noncommutative WZW theory,” referred to in [8, §5] and summarized in part in the survey articles [11] and [34]. Most of this literature seems to deal with the Wess-Zumino-Witten model (where spacetime is a compact group) or with the Moyal product, but we have been unable to find anything that applies to our situation where both spacetime and the worldsheet are represented by noncommutative  $C^*$ -algebras (or dense subalgebras thereof). For that reason, we will attempt here to reformulate the theory from scratch.

The classical Wess-Zumino term is associated to a closed 3-form  $H$  with integral periods on  $X$  (the spacetime manifold). If  $\Sigma^2$  is the boundary of a 3-manifold  $W^3$ , and if  $\varphi: \Sigma \rightarrow X$  extends to  $\tilde{\varphi}: W \rightarrow X$ , the Wess-Zumino term is

$$\mathcal{L}_{WZ}(\varphi) = \int_W (\tilde{\varphi})^*(H).$$

The fact that  $H$  has integral periods guarantees that  $e^{2\pi i \mathcal{L}_{WZ}(\varphi)}$  is well-defined, i.e., independent of the choice of  $W$  and the extension  $\tilde{\varphi}$  of  $\varphi$ .

To generalize this to the noncommutative world, we need to dualize all spaces and maps. We replace  $X$  by  $\mathcal{B}$  (which in the classical case would be  $C_0(X)$ ),  $\Sigma$  by  $\mathcal{A}$ , and  $W$  by  $\mathcal{C}$ . Since  $H$  classically was a cochain on  $X$  (for de Rham

cohomology), it becomes an odd *cyclic cycle* on  $\mathcal{B}$ . The integral period condition can be replaced by requiring

$$(16) \quad \langle H, u \rangle \in \mathbb{Z}$$

for all classes  $u \in K^1(\mathcal{B})$  (dual  $K$ -theory, defined via spectral triples or some similar theory). The inclusion  $\Sigma \hookrightarrow W$  dualizes to a map  $q: \mathcal{C} \rightarrow \mathcal{A}$ , and we suppose  $\varphi: \mathcal{B} \rightarrow \mathcal{A}$  has a factorization

$$\begin{array}{ccc} & & \mathcal{C} \\ & \tilde{\varphi} \nearrow & \downarrow q \\ \mathcal{B} & \xrightarrow{\varphi} & \mathcal{A} \end{array}$$

The noncommutative Wess-Zumino term then becomes

$$\mathcal{L}_{WZ}(\varphi) = \langle \tilde{\varphi}_*(H), [\mathcal{C}] \rangle,$$

with  $[\mathcal{C}]$  a cyclic cochain corresponding to integration over  $W$ . The integral period condition is relevant for the same reason as in the classical case—if we have another “boundary” map  $q': \mathcal{C}' \rightarrow \mathcal{A}$  and corresponding  $\tilde{\varphi}': \mathcal{B} \rightarrow \mathcal{C}'$ , and if  $\mathcal{C} \oplus_{\mathcal{A}} \mathcal{C}'$  is “closed,” so that  $[\mathcal{C}] - [\mathcal{C}']$  corresponds to a class  $u \in K^1(\mathcal{C} \oplus_{\mathcal{A}} \mathcal{C}')$ , then

$$\langle \tilde{\varphi}_*(H), [\mathcal{C}] \rangle - \langle \tilde{\varphi}'_*(H), [\mathcal{C}'] \rangle = \langle H, (\tilde{\varphi} \oplus \tilde{\varphi}')^*(u) \rangle \in \mathbb{Z},$$

and thus  $e^{2\pi i \mathcal{L}_{WZ}(\varphi)}$  is the same whether computed via  $[\mathcal{C}]$  or via  $[\mathcal{C}']$ .

Now we want to apply this theory when  $\mathcal{A} = A_\theta$  (or a suitable smooth subalgebra, say  $A_\theta^\infty$ ). If we realize  $A_\theta$  as the crossed product  $C^\infty(S^1) \rtimes_\theta \mathbb{Z}$ , we can view  $A_\theta^\infty$  as the “boundary” of  $\mathcal{C} = C^\infty(D^2) \rtimes_\theta \mathbb{Z}$ , where  $D^2$  denotes the unit disk in  $\mathbb{C}$ . The natural element  $[\mathcal{C}]$  is the trace on  $\mathcal{C}$  coming from normalized Lebesgue measure on  $D^2$ .

To summarize, it is possible to enhance the sigma-model action on a spacetime algebra  $\mathcal{B}$  with the addition of a Wess-Zumino term  $\mathcal{L}_{WZ}(\varphi)$ , depending on a choice of a “flux”  $H$ .

**4.3. More general spacetimes.** In references such as [22, 23, 24], T-duality considerations suggested that very often one should consider spacetimes which are not just noncommutative tori, but “bundles” of noncommutative tori over some base space, such as the  $C^*$ -algebra of the discrete Heisenberg group, called the “rotation algebra” in [1]. A theory of some of these bundles was developed in [12].

For present purposes, the following definition will suffice:

*Definition 4.1.* Let  $Z$  be a compact space and let  $\Theta: Z \rightarrow \mathbb{T}$  be a continuous function from  $Z$  to the circle group. We define the *noncommutative torus bundle algebra* associated to  $(Z, \Theta)$  to be the universal  $C^*$ -algebra  $A = A(Z, \Theta)$  generated over a central copy of  $C(Z)$  (continuous functions vanishing on the base space,  $Z$ ) by two unitaries  $u$  and  $v$ , which can be thought of as continuous functions from  $Z$  to the unitaries on a fixed Hilbert space  $\mathcal{H}$ , satisfying the commutation rule

$$(17) \quad u(z)v(z) = \Theta(z)v(z)u(z).$$

Note that  $A$  is the algebra  $\Gamma(Z, \mathcal{E})$  of sections of a continuous field  $\mathcal{E}$  of rotation algebras, with fiber  $A_{\log \Theta(z)/(2\pi i)}$  over  $z \in Z$ .

*Examples 4.2.* The reader should keep in mind three key examples of Definition 4.1. If  $Z = \{z\}$  is a point,  $A(Z, \Theta)$  is just the rotation algebra  $A_{\log \Theta(z)/(2\pi i)}$ . More generally, if  $\Theta$  is a constant function with constant value  $e^{2\pi i \theta}$ , then  $A(Z, \Theta) = C(Z) \otimes A_\theta$ . And finally, there is a key example with a nontrivial function  $\Theta$ , that already came up in [23] from T-dualization of  $\mathbb{T}^3$  (viewed as a principal  $\mathbb{T}^2$ -bundle over  $\mathbb{T}$ ) with a nontrivial H-flux, namely the group  $C^*$ -algebra of the integral Heisenberg group. In this example,  $Z = S^1 = \mathbb{T}$  and  $\Theta: \mathbb{T} \rightarrow \mathbb{T}$  is the identity map. If  $w$  is the canonical unitary generator of  $C(Z)$ , then in this case the commutation rule (17) becomes simply  $uv = wvw$  (with  $w$  central), so as explained in [1],  $A$  is the universal  $C^*$ -algebra on three unitaries  $u, v, w$ , satisfying this commutation rule.

*Remark 4.3.* Let  $A = A(Z, \Theta)$  be as in Definition 4.1, and fix  $\theta$  irrational. Then homomorphisms  $A \rightarrow A_\theta$ , not assumed necessarily to be unital, can be identified with triples consisting of the following:

- (1) a projection  $p \in A_\theta$  which represents the image of  $1 \in A$ ,
- (2) a unital  $*$ -homomorphism  $\rho$  from  $C(Z)$  to  $pA_\theta p$ , and
- (3) a unitary representation of the Heisenberg commutation relations (17) into the unital  $C^*$ -algebra  $pA_\theta p$ , with the images of  $u$  and  $v$  commuting with  $\rho(C(Z))$ .

Even in the case discussed above with  $A = C^*(u, v, w \mid uv = wvw)$  and in the special case of unital maps, the classification of maps  $\varphi: A \rightarrow A_\theta$  is remarkably intricate. For example, choose any  $n$  mutually orthogonal self-adjoint projections  $p_1, \dots, p_n$  in  $A_\theta$  with  $p_1 + \dots + p_n = 1$ . Each  $p_j A_\theta p_j$  is Morita equivalent to  $A_\theta$ , and is thus isomorphic to a matrix algebra  $M_{n_j}(A_{\theta_j})$ ,  $\theta_j \in GL(2, \mathbb{Z}) \cdot \theta$ . For each  $j$ , there is a unital map  $\varphi_j: A \rightarrow M_{n_j}(A_{\theta_j})$  sending the central unitary  $w$  to  $e^{2\pi i \theta_j}$ . Then  $\varphi_1 \oplus \dots \oplus \varphi_n$  is a unital  $*$ -homomorphism from  $A$  to  $A_\theta$  sending  $w$  to  $\sum e^{2\pi i \theta_j} p_j$ . Since  $n$  can be chosen arbitrarily large, one sees that there are quite a lot of inequivalent maps. In this particular example,  $K_1(A)$  is a free abelian group on 3 generators,  $u, v$ , and an additional generator  $W \in M_2(A)$  [1, Proposition 1.4] ( $w$  does not give an independent element since it is the commutator of  $u$  and  $v$ ). A notion of “energy” for such maps  $\varphi$  may be obtained by summing the energies of the three unitaries  $\varphi(u)$ ,  $\varphi(v)$ , and  $\varphi(W)$  (for the last of these, one needs to extend  $\varphi$  to matrices over  $A$  in the usual way). Estimates for the energy can again be obtained using the results and methods of [20].

## 5. A PHYSICAL MODEL

To write the partition function for the sigma-model studied in this paper, recall the expression for the energy from equation (15),

$$\mathcal{L}_{G,D}(\varphi) = \varphi^*(\psi_2)(G) = \sqrt{\det(g)} \sum_{j,k=1}^2 \sum_{\mu,\nu=1}^2 G_{ij} g^{\mu\nu} \text{Tr}(\delta_\mu(\varphi(U_j))^* \delta_\nu(\varphi(U_k))).$$

It is possible to parametrize the metrics  $(g_{\mu\nu})$  by a complex parameter  $\tau$ ,

$$g(\tau) = (g_{\mu\nu}(\tau)) = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}$$

where  $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$  is such that  $\tau_2 > 0$ . Note that  $g$  is invertible with inverse given by

$$g^{-1}(\tau) = (g^{\mu\nu}(\tau)) = \tau_2^{-2} \begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix}$$

and  $\sqrt{\det(g)} = \tau_2$ . The ‘‘genus 1’’ partition function is

$$Z(G, z) = \int_{\tau \in \mathbb{C}, \tau_2 > 0} \frac{d\tau \wedge d\bar{\tau}}{\tau_2^2} Z(G, \tau, z)$$

where

$$Z(G, \tau, z) = \int \mathcal{D}[\varphi] e^{-z\mathcal{L}_{G,\tau}(\varphi)} / \int \mathcal{D}[\varphi].$$

is the renormalized integral. Here  $\mathcal{L}_{G,\tau} = \mathcal{L}_{G,D}$ , where we emphasize the dependence of the energy on  $\tau$ . This integral is much too difficult to deal with even in the commutative case, so we oversimplify by considering the semiclassical approximation, which is a sum over the critical points. Even this turns out to be highly nontrivial, and we discuss it below. In the special case when  $\Theta = \theta$  and is not a quadratic irrational, then the semiclassical approximation to the partition function above is

$$Z(G, \tau, z) \approx \sum_{m \in M/\{\pm 1\}} \sum_A e^{-z\mathcal{L}_{G,\tau}(\varphi_A)},$$

up to a normalizing factor, in the notation as explained later in this section. In this approximation,

$$Z(G, z) \approx \int_{\tau \in \mathbb{C}, \tau_2 > 0} \frac{d\tau \wedge d\bar{\tau}}{\tau_2^2} \sum_{m \in M/\{\pm 1\}} \sum_A e^{-z\mathcal{L}_{G,\tau}(\varphi_A)}.$$

We expect  $Z(G)$  and  $Z(G^{-1})$  to be related as in the classical case [27, 5], as a manifestation of T-duality.

In the rest of this section we specialize to a (rather oversimplified) special case based on the results of Section 3.1. As explained before, we basically take our spacetime to be a noncommutative 2-torus, and for simplicity, we ignore the integral over  $\tau$  (the parameter for the metric on the worldsheet) and take  $\tau = i$ .

As pointed out by Schwarz [33], changing a noncommutative torus to a Morita equivalent noncommutative torus in many cases amounts to an application of T-duality, and should not change the underlying physics. For that reason, it is perhaps appropriate to stabilize and take our spacetime to be represented by the algebra  $A_\Theta \otimes \mathcal{K}$  ( $\mathcal{K}$  as usual denoting the algebra of compact operators), which encodes all noncommutative tori Morita equivalent to  $A_\Theta$  at once. (Recall  $A_{\Theta'}$  is Morita equivalent to  $A_\Theta$  if and only if they become isomorphic after tensoring with  $\mathcal{K}$ , by the Brown-Green-Rieffel theorem [4].)

Since this algebra is stable, to obtain maps into the worldsheet algebras we should take the latter to be stable also, and thus we consider a sigma-model based on maps  $\varphi: A_\Theta \otimes \mathcal{K} \rightarrow A_\theta \otimes \mathcal{K}$ , where  $\theta$  is allowed to vary (but  $\Theta$  remains fixed). Via the results of Section 2, such maps exist precisely when there is a morphism of ordered abelian subgroups of  $\mathbb{R}$ , from  $\mathbb{Z} + \mathbb{Z}\Theta$  to  $\mathbb{Z} + \mathbb{Z}\theta$ , or when there exists  $c\theta + d \in \mathbb{Z} + \mathbb{Z}\theta$ ,  $c\theta + d > 0$ , such that  $(c\theta + d)\Theta \in \mathbb{Z} + \mathbb{Z}\theta$ , i.e., when there exists



$m \in M = GL(2, \mathbb{Q}) \cap M_2(\mathbb{Z})$  (satisfying the sign condition  $c\theta + d > 0$ ) such that  $\Theta = m \cdot \theta$  or  $\theta = m^{-1} \cdot \Theta$  for the action of  $GL(2, \mathbb{Q})$  on  $\mathbb{R}$  by linear fractional transformations.

Given that  $\Theta = m \cdot \theta$  for some  $m \in M$ , the matrix  $m$  determines the map  $\varphi_*: K_0(A_\Theta \otimes \mathcal{K}) \rightarrow K_0(A_\theta \otimes \mathcal{K})$ , which turns out to be multiplication by

$$(18) \quad \mathcal{D}(m, \theta) = |c\theta + d| \text{ if } m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(Note the similarity with the factor that appears in the transformation law for modular forms. Also note that if  $\Theta = m \cdot \theta$ , then also  $\Theta = m' \cdot \theta$  for many other matrices  $m'$ , since one can multiply both rows by the same positive constant factor.) However,  $m$  does not determine the map induced by  $\varphi$  on  $K_1$ . A natural generalization of Conjecture 3.1 would suggest that if  $\theta = \Theta$  and  $m = 1$ , at least if  $\theta$  is not a quadratic irrational (so as to exclude the Kodaka-like maps), then the induced map  $\varphi_*$  on  $K_1$  has a matrix

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

in  $SL(2, \mathbb{Z})$ , and there should be (up to gauge equivalence) a unique energy-minimizing map  $\varphi: A_\Theta \otimes \mathcal{K} \rightarrow A_\theta \otimes \mathcal{K}$  with energy

$$4\pi^2 (p^2 + q^2 + r^2 + s^2).$$

Note that  $p^2 + q^2 + r^2 + s^2$  is the squared Hilbert-Schmidt norm of  $A$  (i.e., the sum of the squares of the entries). We want to generalize this to the case of other values of  $m$ .

Unfortunately, the calculation in Section 3.3 suggests that there may not be a good formula for the energy of a harmonic map just in terms of the induced maps on  $K_0$  and  $K_1$ . But a rough approximation to the partition function might be something like

$$(19) \quad Z(z) \approx \sum_{m \in M/\{\pm 1\}} \sum_A e^{-4\pi^2 \mathcal{D}(m, \theta) \|A\|_{HS}^2 z}.$$

The formula  $4\pi^2 \mathcal{D}(m, \theta) \|A\|_{HS}^2$  for the energy is valid not just for the automorphisms  $\varphi_A$  but also for the map  $U \mapsto u^p v^q$ ,  $V \mapsto u^r v^s$  with

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad \det A = n$$

from  $A_{n\theta}$  to  $A_\theta$ , which one can check to be harmonic, just as in Corollary 3.10. The associated map on  $K_0$  corresponds to the matrix

$$m = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$$

with  $\mathcal{D}(m, \theta) = 1$ .

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