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THE UNIVERSITY OF PRETORIA

FIRST SEMESTER, 2012
Campus: Hatfield

PHYSICS 255

Modern Physics
Second Semester Test
Total: 80

(Time allowed: TWO hours)

Internal Examiner: M. van den Worm

External Examiner: Q. Odendaal

NOTE:
- This question paper consists of 7 pages, 5 sections (A - E) and 13 questions.
- Answer all the questions.
- Use correct notation, indicate vectors by $\overline{x} = (a, b, c)$.
- When deriving formulas indicate clearly what you are doing.
- Try and work neat, it is easier to grade a neat paper.
- DO NOT TURN THE PAGE

SECTION A

The Bohr Model

1. Consider an electron orbiting in a circular path around a nucleus with Z protons. Only consider forces due to the interaction between the electron and the nucleus.

(a) Show that $mv^2 = \frac{Zke^2}{r^2}$.

(b) Show that we can write the total energy of the electron as $E = -\frac{1}{2}\frac{Zke^2}{r}$.

(c) Given the fact that angular momenta is quantized in integer multiples of $\hbar$, show that the values of the allowed radii are quantized by $r = n^2 \frac{a_B}{Z}$ where $a_B = \frac{\hbar^2}{ke^2 m}$.

(d) Show that the Bohr model finds the same form as that of the Rydberg formula for the energies of the emitted photons $E_\gamma = Z^2 hcR\left(\frac{1}{n'^2} - \frac{1}{n^2}\right)$ where $R = \frac{ke^2}{2a_B hc}$.

(8 marks)

2. (a) What are the nergy and wabelength of photons in the Lyman α line of $\mathrm{Fe}^2 5+$?

(b) What kind of electromagnetic radiation is this?

(2 marks)

SECTION B

Matter Waves

3. According to the Born interpretation of quantum mechanics, what is a matter wave? (2 marks)

4. Let $F(x)$ be a periodic function that is odd; that is $F(x) = -F(-x)$. The Fourier expansion of sucha function requires only sine functions:

$$F(x) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n\pi x}{\lambda}\right) \tag{1}$$

where λ is the period (or wavelength) of the function.

(a) Prove that

$$B_m = \frac{2}{\lambda}\int_0^{\lambda} F(x)\sin\left(\frac{2m\pi x}{\lambda}\right)dx. \tag{2}$$

(b) Use this result to show what the Fourier coefficients B_n of the "sawtooth" function are zero for n even and

$$B_n = (-1)^{(n-1)/2}\frac{8}{\pi^2 n^2} \tag{3}$$

for n odd.

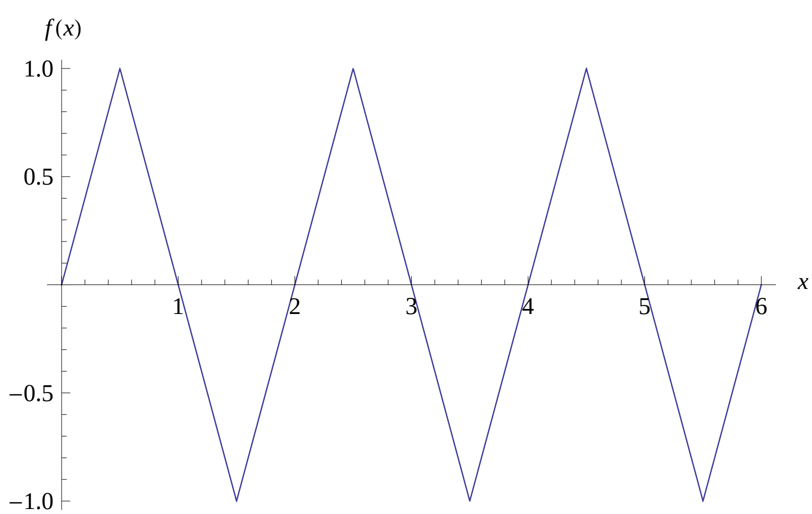

Figure 1: The "sawtooth" function

(8 marks)

5. (a) Write down the Heisenberg uncertainty relation for position and momentum

(b) An electron is known to be somewhere in an interval of total width $a \approx 0.1\text{nm}$. What is the minimum uncertainty in its velocity?

(c) Show that the minimum kinetic energy of this electron is given by

$$\langle K \rangle \gtrsim \frac{\hbar^2}{2ma^2} \tag{4}$$

and calculate $\langle K \rangle$.

(d) Write down the Heisenberg uncertainty relation for time and energy?

(e) Using $\langle K \rangle$ as the uncertainty in energy, what will be the uncertainty in time?

(10 marks)

SECTION C

Schrödinger Equation Problems

6. (a) Write down the one dimentional time dependant Schrödinger Equation.

(b) Show that for standing wave solutions $\Psi(x,t) = \psi(x)e^{-i\omega t}$ the spatial wave function $\psi(x)$ satisfies the one dimensional time independant Schrödinger equation.

(5 marks)

7. (a) Show that the wave function of a particle trapped in the one dimentional rigid box

$$\psi(x) = A\sin(\frac{n\pi x}{a}) \tag{5}$$

can by found by solving the Schödinger equation with a potential

$$U(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \\ \infty & \text{for } x \leq 0 \text{ and } x \geq a \end{cases} \tag{6}$$

(b) Calculate the amplitude of the wave function.

(c) Determine the energy as a function of n.

(d) Calculate the expectation value of position $\langle x \rangle$ for the ground state of the particle trapped in the one dimentional rigid box.

(10 marks)

8. (a) Using either the determinant method or separation of variables show that the wave function of a particle trapped inside a two dimentional rigid box with potential

$$U(x,y) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \text{ and } 0 \leq y \leq b \\ \infty & \text{otherwise} \end{cases} \tag{7}$$

is given by

$$\psi_{m,n}(x,y) = A\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{8}$$

(b) Determine the quantized energy as a function of m and n.

(c) If we assume $a = b$ we will have degeneracy in the energies. What is the physical significance of this?

(d) How can such degeneracies by removed in more realistic models?

(10 marks)

SECTION D

Hydrogen Atom

9. Consider the ground state of the Hydrogen atom. We know that the wave function should be spherically symmetric and has no angular dependance. The radial wave function is given by

$$\frac{d^2}{dr^2}(rR) = \frac{2m_e}{\hbar^2}\left(-\frac{ke^2}{r} + \frac{E_R}{n^2}\right)(rR) \tag{9}$$

(a) Show that

$$R_{1s}(r) = Ae^{-r/a_B} \tag{10}$$

is a solution of the radial Schrödinger equation with $a_B = \frac{\hbar^2}{m_e^2 ke^2}$ the Bohr radius and $E_R = \frac{ke^2}{2a_B}$ the Rydberg energy.

(b) Show that we can write the radial probability density as (give reasons)

$$P_{1s}(r) = 4\pi r^2 |R_{1s}(r)|^2 \tag{11}$$

(c) Show that we have the greatest probability of finding the electron at $r = a_B$

(6 marks)

10. (a) Draw digrams of the radial probability distributions for the 1s, 2s, 2p, 3s, 3p and 3d states.

(b) From your diagrams comment on the validity of the shell model of the Hydrogen atom.

(4 marks)

SECTION E

Spin and Pauli Exclusion

11. Consider the classical point electron traveling around a nucleus in a circular orbit in the presence of an external magnetic field. This orbiting charge acts like a small current loop

(a) Give an expression for the torque experienced by the current loop. Let $\overline{A}$ denote the area vector and $\overline{B}$ the magnetic field.

(b) If we define $\overline{\mu} := i\overline{A}$ where i is the current in the current loop. Show that the work done by the magnetic field to bring the loop to an angle θ is given by

$$W = \mu B \cos(\theta) + \text{constant} \tag{12}$$

(c) From the above show that we have a potential

$$U = -\overline{\mu}.\overline{B} \tag{13}$$

(d) Given that we can write the angular momentum of an orbiting electron as $L = m_e vr$. Show that $\overline{\mu} = -\frac{e}{2m_e}\overline{L}$

(e) If the magnetic field is applied along the z-axis what will be the shift in energy of the electron?

(10 marks)

12. Consider a He_2 atom in one of its singlet states. One electron is in an s state and the other in a d state (l=2). The atom is placed in a magnetic field of $B = 2$T. By how much does the magnetic field change the atom's energy? (Hint: $\mu_B = 5.79 \times 10^{-5}$eV/T) (2 marks)

13. (a) State the Pauli Exclusion Principle.

(b) Draw a tree diagram showing all the allowed quantum numbers for the 2p state of an electron and comment on the degeneracy.

(3 marks)

-THE END-